FUNDAMENTAL THEOREM OF COMPLEX POLYNOMIALS BY MEANS OF AN ITERATED REAL INTEGRAL

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Abstract

It is understood that in proving that any complex polynomial of one variable must possess a root, some part of the argument must involve topology or analysis: the proof needs an analytic definition of "complex numbers". Historical treatments of this "Fundamental theorem of complex polynomials" considered the case of "all real" coefficients (but the root to be found might be part imaginary). The generalization to "complex coefficients" was seen to follow immediately (by unique factorization), around 1850. Nowadays everybody refers to this theorem as "Fundamental Theorem of Algebra" (FTA). In fact the case of odd degree (of polynomial) comes from a basic result (IVP) on *real* numbers. In this note we deal with the remaining case, a complex polynomial of *even* degree greater than zero, whose values as a function on the *real numbers* remain *real*. The main tools are 1) Cauchy-Riemann (differentiability) property of the polynomial, 2) majorization of the polynomial by a term of the same degree, 3) reversibility of the order of integration, of an iterated integral involving two independent variables. Note that there are no "multiple" or "multi-dimensional" integrals involved, though the first integral of our real (two-variable) function leads to a (one-variable) function. The proof (which only uses rectilinear Riemann or "Jordan" integrals) is related to known proofs based on Cauchy's Integral Theorem, which however also employ transcendental functions. The main technical tool for our proof is validity in the proper context of "differentiation under the integral sign". The connection of this result to Clairaut's Theorem on reversed order of differentiation is explained, together with an alternative treatment based on explicit Weierstraß approximation of a rational function.

Introduction

Any proof of the "Fundamental Theorem of Algebra" (FTA), that a polynomial of degree ≥ 1, over the complex numbers must have a root, contains an analytical argument. Such a proof makes use of the topology (connectedness, completeness) of the real members $\mathbb{R}$ or the complexes $\mathbb{C}$, which are to start with, infinite sets. We offer such a proof that uses only the simplest properties of the Riemann integral over a finite segment in the plane $\mathbb{R}^2$, applied to a continuous function of two real variables. In fact, we could use the "Jordan Integral", where all partitions of the

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segment are made to have equally-spaced division points informing the Riemann sum and its limit. Furthermore, only rational expression, quotients $p(x, y)/q(x, y)$ actually come up, where p, q are real polynomials (multi-nomials) of two variables.

This method is related to approaches from [Boas], [Ankeny] or [Burkel]. It does not rely on contour integration or two-dimensional "measure and integration" per se. Nor does it invoke the polar argument, exponential/trigonometric functions, extraction of roots, fixed-points or mean values, closed or open or proper mappings. We do exploit the one-dimensional Riemann or Jordan integral and its textbook properties, the "growth inequality" for a complex polynomial, the Cauchy-Riemann property (derived herein), derivative properties of a rational function, and uniform continuity.

In an alternative proof concerning iterated integrals, we approximate a continuous function by a explicit multi-nomial.

The principal proposed proof does not seem to be suitable for generalizations of FTA such as: a function on $\mathbb{C}$ given by $g(z) = z^n + z^n \cdot \xi(z)$, $n > 0$, when $\xi(z)$ is continuous and goes to 0 as $|z| \to \infty$, must have a root z^*, so $g(z^*) = 0$. We derive the classical FTA for a polynomial $P(z)$ aided by the Cauchy-Riemann property of $P(x + iy)$.

It is well-known, see previous references, that to prove FTA it is sufficient to consider those $P(z)$ that take real values for $z = x + i \cdot 0$. Also limiting ourselves to a *monic* polynomial P, we note that this "reality" condition on the x-axis cannot hold (due to the Intermediate Value Theorem) when the degree n is odd, unless zeros $\{x^*\}$ crop up on said x-axis, so we may take $n = 2m$.

We have fallen back on the problem of finding a root for

$$P(z) = z^{2m} + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 \ , \tag{1}$$

where $a_0 \neq 0$, $a_1, \dots, a_{n-1} = a_{2m-1}$ are all *real*.

Cauchy-Riemann Equations and a Rational Function Defined on $\mathbb{C}$

A complex valued continuous functions $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ is *differentiable* at a given $x_0 + iy_0 \in \mathbb{C}$ if $\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0)$ and $\frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0)$.

For partial derivatives of $g(x, y)$ we also write $\frac{\partial}{\partial x} g = D_x g$ or $g_x(x, y)$ for example. One way to express the C-R equations in a single formula is by $D_y f(x + iy) = iD_x f(x + iy)$.

We will appeal to the elementary properties of real rational function under integration. The facts about the class of rational functions of one (resp. two) real variables are similar to the facts about continuous functions. Assuming that g and h are continuous on Ω, an open subset of $\mathbb{R}$ or $\mathbb{R}^2$, or the closure of such an open set, then

(i) a linear combination $\alpha g + \beta h$, $\alpha, \beta \in \mathbb{R}$, is continuous,
(ii) the product gh is continuous,
(iii) as long as $g(x) \neq 0$ (resp. $g(x, y) \neq 0$) on Ω, the reciprocal $\dfrac{1}{g(x)}$ is also continuous on Ω.

The above statement with the numbered conclusions holds equally well when we replace "continuous" with "rational" throughout. On a closed domain such as

$J \subset \mathbb{R}^2$, $J = [a,b] \times [c,d]$, we may consider "one-sided continuity" on boundary points.

In particular, since constant functions and the monomials x and y are continuous, all real polynomials $g(x)$ or $w(x,y)$ in one or two variables are continuous on the type of domain Ω that we are considering. Also a rational function $f(x,y) = \dfrac{1}{w(x,y)}$ is continuous on Ω where $w(x,y)$ never vanishes.

Under these same conditions we easily see that the Cauchy-Riemann conditions hold. The only challenge is to deduce from "$w(x+iy)$ is a C-R function" the statement "so is $f(x+iy) = \dfrac{1}{w(x+iy)} = u(x,y) + iv(x,y)$". This is true because

$$D_y f = D_y \left(\frac{1}{w}\right) = \frac{-D_y w}{w^2(x,y)} = \frac{-iD_x w}{w^2} = iD_x \left(\frac{1}{w}\right) \ ,$$

where at the middle step we used Cauchy-Riemann for $w(x,y)$. Note that if we write $w(x+iy)$ as $\sigma(x,y) + i\tau(x,y)$, we see

$$f(x+iy) = \frac{\sigma(x,y) - i\tau(x,y)}{\sigma^2 + \tau^2} \ ,$$

which expresses $u(x,y)$ and $v(x,y)$ as rational functions. Now we may let $w(z) = w(x+iy) = P(z)$.

Since powers of z and hence polynomials satisfy the C-R condition and there obtain

$$\begin{aligned} u_x(x,y) &= v_y(x,y) \\ u_y(x,y) &= -v_x(x,y) \ , \end{aligned} \tag{2}$$

where $f(x+iy)$ is defined. If we assume that $\sigma^2(x,y) + \tau^2(x,y)$ has no zeros on Ω (and eventually not on $\mathbb{R}^2$ either), the Cauchy-Riemann functions u and v (actually *harmonic conjugates*) are defined, rational and so continuous and differentiable to all orders on Ω.

Linear Calculus on a Rational Function of Two Variables

The basic properties of the definite integral of a single-variable function are listed in handbooks such as [Bronshtein], p. 263-265, and texts such as [Rosenlicht], p. 116-118 and p. 126. These rules apply to Riemann (or Jordan) integrable functions, but our application will involve only continuous (in fact *uniformly continuous*) functions.

(0) Zero Integral: $\displaystyle\int_a^a f(x)dx = 0$. Also if g is the zero function on $[a,b]$, we have $\int_a^b g(x)dx = 0$. If $g(x)$ is continuous and *not* the zero function on $[a,b]$, then there is a subinterval $[a',b']$ with $a \le a' < b' \le b$ such that $\int_{a'}^{b'} g(x)dx \neq 0$.

(1) Linearity: for constants $\alpha, \beta \in \mathbb{R}$ we have

$$\int_a^b [\alpha f(x) + \beta g(x)]\, dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx\,.$$

(2) Additivity over the domain: given $a < c < b$, we have

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx\,.$$

(3) Majorization: $f(x) \le g(x)$ on $[a, b]$, $a < b$, implies

$$\int_a^b f(x)dx \le \int_a^b g(x)\,.$$

Hence for example, $f(x) \le M$ implies

$$\int_a^b f(x)dx \le (b - a)M\,.$$

(4) Triangle Inequality:

$$\left| \int_a^b f(x)dx \right| \le \int_a^b |f(x)|dx\,.$$

(5) Fundamental Theorem of the Calculus for one-dimensional Riemann (or Jordan) integrals: if $f(x)$ is continuous on the interval $[a, b]$ and $f(x) = \mathrm{G}'(x)$ there, then

$$\int_a^b f(x)dx = \mathrm{G}(b) - \mathrm{G}(a)\,.$$

Also if $f(t)$ is continuous for $a < t < b$, then

$$F(t) := \int_a^t f(x)dx$$

is differentiable with $\dfrac{dF(t)}{dt} = f(t)$.

More advanced results involving several variables will be recapitulated. Since we work with iterated integrals, it is useful to see that the integration one time of a continuous function (of two variables) yields a continuous function.

Lemma 1. Let $f : J \to \mathbb{R}$ be continuous on $J = [a, b] \times [c, d]$ (or on an open neighborhood of J). Then defining $g : [a, b] \to \mathbb{R}$ by

$$g(x) = \int_c^d f(x, y)dy \ ,$$

the function $g(x)$ will be continuous on $I_1 = [a, b]$.

Proof. Choosing $a \le x_0 < x_1 \le b$, we have

$$|g(x_1) - g(x_0)| \le \int_c^d |f(x_1, y) - f(x_0, y)| \, dy$$

by Point 4, the Triangle Inequality for integrals. Since $f(x, y)$ is uniformly continuous on compact J, one may choose $\delta > 0$ small enough that the implication holds:

$$|x_1 - x_0| < \delta \quad \Rightarrow \quad |f(x_1, y) - f(x_0, y)| < \frac{\epsilon}{d - c} \ ,$$

for a pre-assigned $\epsilon > 0$. By the Majorization rule, Point 3, we obtain $|g(x_1) - g(x_0)| < \epsilon$; thus $g(x)$ is actually *uniformly* continuous on $I_1 = [a, b]$. ∎

We will be considering rational functions (hence continuous) of two variables well-defined on J. The boundary rectangle $B = \partial J$ consists of two ordered intervals of length $b - a$ parallel to the x-axis, unioned with two closed intervals of length $d - c$ parallel to the y-axis. In the more general case of an iterated integrals for *continuous* functions, in fact we have

Proposition 1. "Equality of Reversed Iterated Integrals". For $g(x, y)$ a continuous function well-defined on $\Omega \subset \mathbb{R}^2$, an open set containing $J = [a, b] \times [c, d]$, there holds

$$\int_c^d \int_a^b g(x, y) \, dxdy = \int_a^b \int_c^d g(x, y) \, dydx \ .$$

Before proving this Proposition, we state

Lemma 2. "Differentiation under the Integral Sign". If $g(x, y)$ is continuously differentiable on J, that is $g \in C^1(J)$, consider

$$\mathrm{G}(u) = \int_a^b g(x, u) \, dx$$

where $c < u < d$. Since $g \in C^1(J)$, the partial derivative $\dfrac{\partial g}{\partial y}(x, u)$ is continuous for J. We conclude that $\mathrm{G}'(u)$ exists and equals

$$\int_a^b \frac{\partial g}{\partial y}(x, u) \, dx \ ,$$

which is continuous in u (by Lemma 1).

Proof of Lemma 2. The hypotheses imply that $\dfrac{\partial g}{\partial y} = g_y(x,u)$ is uniformly continuous on compact J. We may set

$$\Delta\mathrm{G}(y) = \int_a^b \left[g\left(x, y+\Delta y\right) - g(x,y)\right] dx\ ,$$

so that $|\Delta\mathrm{G}(y)| < (b-a)\epsilon$ where $|\Delta y| < \delta(\epsilon)$, δ chosen independently of y in $[c,d]$. This argument already demonstrates that $\mathrm{G}(u)$ is (uniformly) continuous on $[c,d]$, (at the boundary points $u=c$, $u=d$ we may work with the appropriate one-sided continuity).

Similarly, $g_y(x,y)$ is approximated by $\dfrac{1}{\Delta y}\left[g(x,y+\Delta y) - g(x,y)\right]$. Thus an increment, possibly smaller than before, $\delta > 0$ can be chosen so that

$$\left|\frac{\left[g\left(x, y+\Delta y\right) - g(x,y)\right]}{\Delta y} - \frac{\partial g}{\partial y}(x,y)\right| < \epsilon \quad \text{for} \quad (x\,,y) \in\ J\ .$$

If we integrate the terms of the left-hand side (L.H.S) of this inequality, and apply Majorization, Point 3, we obtain

$$\left|\frac{\Delta\mathrm{G}(u)}{\Delta y} - \int_a^b \frac{\partial g(x,u)}{\partial y}\, dx\right| < \epsilon(b-a)\ ,$$

for any $\Delta y < \delta$.

By definition of the limit, as $\Delta y \to 0$, the quantity $\Delta\mathrm{G}(u)/\Delta y$ approaches $\mathrm{G}'(u)$ as well as the real number

$$\int_a^b \frac{\partial}{\partial y}\, g(x,u)\, dx\ .$$

Thus we have proved the Differentiation formula for $\mathrm{G}'(u)$, $u \in I_2 = [c,d]$. Also we have exhibited this (derivative) function as continuous by using Lemma 1, with the second variable u as independent variable. Since $g_y(x,u)$ is continuous for $(x,u) \in J$, its integral $\mathrm{G}'(u)$ is continuous as well for $u \in [c,d]$. ■

We now complete the proof of Proposition 1.

Define where $g(x,y)$ is continuous,

$$r(t) = \int_a^t \int_c^d g(x,y)\, dy dx$$
$$s(t) = \int_c^d \int_a^t g(x,y)\, dx dy\ .$$

These functions are continuous in one variable $t \in [a,b]$. By Point 5, the Fundamental Theorem of Calculus, we have

$$r'(t) = \int_c^d g(t,y)\, dy\ .$$

On the other hand, by Lemma 2 we may apply differentiation to $s(t)$ obtaining

$$s'(t) = \int_c^d g(t,y)\,dy\ .$$

These two continuous functions of t have the same values on the interval $[a,b]$, hence $r(t)$, $s(t)$ differ by a constant C. Furthermore, $r(t)$ and $s(t)$ are continuously differentiable. But $r(a) = s(a) = 0$, so this constant must equal zero and the "reversed" iterated integrals yield identical functions. ■

Remark. From Proposition 1, it is easy to verify the well-known Schwarz-Clairaut Theorem (Equality of Mixed Partials), that for $g(x,y) \in C^2(\Omega)$, Ω open in $\mathbb{R}^2$, one concludes $g_{xy} = g_{yx}$ on Ω. Of course if g is a rational function, the quotient of two multinomials, this equality may be explicitly checked wherever g is well-defined. In the more general context of C^2 functions, compute

$$(*) \qquad \int_a^x \int_c^y g_{xy}(u,v)\,dvdu = g(x,y) - g(x,c) - g(a,y) + g(a,c)\ .$$

Taking $\dfrac{\partial^2}{\partial x \partial y}$ of the R.H.S. yields $g_{yx}(x,y)$ but working with the L.H.S. gives

$$\frac{\partial}{\partial y} \int_a^x \int_c^y g_{xy}(u,v)\,dvdu = \int_a^x g_{xy}(u,y)\,du$$

by Point 5, the FTC, and by Lemma 2, differentiation under the integral sign. Following up with $\frac{\partial}{\partial x}$ applied to this resulting expression naturally yields $g_{xy}(x,y)$ by the last part of Point 5 (derivative of an integral) within a domain Ω.

Reversal of Integration by Multi-nomial Approximation

Supposing again that $f(x,y)$ is continuous on $J = [a,b] \times [c,d]$, $a < b$, $c < d$, it may be approximated by polynomials in two variables $p_n(x,y) \to f(x,y)$ as $n \to \infty$. That is, given $\epsilon > 0$, there is a natural number $N(\epsilon)$ such that $n > N(\epsilon)$ implies

$$|p_n(x,y) - f(x,y)| < \epsilon \quad \text{on } J.$$

Using Points 1, 3 and 4, Linearity, Majorization and the Triangle Inequality, one obtains

$$\left| \int_c^d \int_a^b p_n(x,y)\,dxdy - \int_c^d \int_a^b f(x,y)\,dxdy \right| = \left| \int_c^d \int_c^b \left(p_n(x,y) - f(x,y)\right)\,dxdy \right| \leq$$

$$\leq \int_c^d \int_a^b |p_n(x,y) - f(x,y)|\,dxdy \leq (b-a)(d-c)\,\epsilon$$

for $n > N(\epsilon)$. Therefore

$$\lim_{n\to\infty} P_n = \lim_{n\to\infty} \int_c^d \int_a^b p_n(x,y)\,dxdy$$

is equal to

$$\int_c^d \int_a^b f(x,y)\,dxdy.$$

For a monomial such as $\phi(x,y) = x^r y^s$, $r, s \geq 0$, the order of iterated definite integration is immaterial, the answer being

$$\frac{(b-a)^{r+1}(d-c)^{s+1}}{(r+1)(s+1)}$$

for either order. It follows that the integrals can also be reversed for any linear combination of monomials, that is, for any multi-nomial $p(x,y)$.

Hence, all the "reversed" values

$$Q_n = \int_a^b \int_c^d p_n(x,y)dydx$$

equal the corresponding P_n, and as above, Q_n converges to

$$\int_a^b \int_c^d f(x,y)\,dydx\ ,$$

so we arrive at a second proof of Proposition 1 (Reversal of Integral), as soon as we exhibit such a multi-nomial sequence $\{p_n(x,y)\}$ that converges (uniformly) on J to our continuous (or rational) function $f(x,y)$. [This fact is of course a version of the Weierstraß Approximation Theorem].

Texts and monographs on Approximation Theory show that either of the following two constructions lead to the sequence of (two-variable) polynomials that is required. See [Davis] or [Burkill].

I. (Bernstein)

$$B_n f(x,y) = \sum_{k,l=0}^{n} \binom{n}{k}\binom{n}{l} f\left(a+\frac{k}{n}(b-a), c+\frac{l}{n}(d-c)\right)\cdot\left(\frac{x-a}{b-a}\right)^k \left(\frac{y-c}{d-c}\right)^l \left(\frac{b-x}{b-a}\right)^{n-k} \left(\frac{d-y}{d-c}\right)^{n-l}.$$

II. (Landau)

Let

$$\rho_n = \frac{2^{n+1}n!}{3\cdot 5\cdot\cdots(2k+1)} = \frac{2\prod_{j=1}^{n}(2j)}{\prod_{j=1}^{n}(2j+1)}.$$

Then

$$L_n f(x,y) = \frac{1}{\rho_n^2 (b-a)(d-c)} \cdot \int_c^d \int_a^b \left[1 - \left(\frac{\alpha - x}{b-a}\right)^2\right]^n \left[1 - \left(\frac{\beta - y}{d-c}\right)^2\right]^n f(\alpha, \beta)\, d\alpha d\beta \; .$$

The above proof of the reversibility of the order of definite integration, by means of uniform multi-nomial approximation, leads to a simple argument for

Theorem (Schwarz-Clairaut). If $f \in C^2(J)$, then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

(or $f_{yx} = f_{xy}$) on J.

Proof sketch. Choosing say the Bernstein sequence for the continuous function $f_{xy}(x,y)$, we obtain

(3) $$\int_c^d \int_a^b B_n(x,y)\, dx dy \to \int_c^d \int_a^b f_{xy}(x,y)\, dx dy = E \; ,$$

where $n \to \infty$.

This number is evaluated by explicit integration as

$$E = f(b,d) - f(b,c) - f(a,d) + f(a,c) \, ,$$

which is also equal to

$$F := \int_a^b \int_c^d f_{yx}(x,y)\, dy dx \; .$$

But we saw how in the case of the two-variable polynomial $B_n(x,y)$, explicit integration shows that an iterated integral can be reversed. So as $n \to \infty$,

$$\int_a^b \int_c^d B_n dy dx$$

goes to F, but also to

$$\int_a^b \int_c^d f_{xy}(x,y)\, dy dx \; ,$$

which gives

(4) $$\int_a^b \int_c^d (f_{xy} - f_{yx})\, dy dx = 0 \; .$$

Since in (4) the integrand is continuous by hypothesis, and the equality holds also for any smaller rectangle $J' \in J$, this integrand must be identically equal to 0 on J, hence $f_{xy}(x,y) = f_{yx}(x,y)$ on our domain. This may be seen by citing Zero Integral, Point 0, for each of the two variables. ∎

An Iterated Integral on J

We observed that the Fundamental Theorem of Complex Polynomials (FTA) could be derived by considering a complex polynomial

$$P(z) = w(x + iy) = z^n + \sum_{j=1}^{n} a_{n-j} z^{n-j} \ ,$$

where $n > 1$ is *even*, and the coefficients $a_n \in \mathbb{R}$, so that $w(x)$ takes only *real*, in fact *positive* values for $x \in \mathbb{R}$. It would follow immediately from the continuity of P, using the Intermediate Value Theorem, that when n is *odd*, even a *real* zero $x_0 \in \mathbb{R}$ must exist, since for x large enough and positive, $P(x)$ attains positive values and $P(-x)$ negative values. See [Redheffer], [Sjogren, Lu]. Real zeros may be ruled out by hypothesis, and we are left with the conclusion that $P(x) > 0$ for all $x \in \mathbb{R}$, as P never changes sign on the x-axis, but certainly attains positive values for x sufficiently large.

An important bound is given in [Hille, vol. I]. Letting $c = \max_k\{|a_k|\}$, the supremum in modulus of the coefficients in $P(z)$ up to a_{k-1}, we write for $|z| > 1$,

$$|P(z)| \geq |z|^n - \left| \sum_{k=1}^{n} a_{n-k} z^{n-k} \right| \geq |z|^n - c \sum_{k=1}^{n} |z|^{n-k} \ .$$

The latter expression equals

$$|z|^n - c\frac{|z|^n - 1}{|z| - 1} > |z|^n \left\{ 1 - \frac{c}{|z| - 1} \right\} \ .$$

It follows that for $|z| > 1 + 2c$, we end up with the "growth inequality"

$$|P(z)| > \frac{1}{2}|z|^n \ . \tag{5}$$

We return to consideration of

$$f(x + iy) = \frac{1}{P(z)} = \frac{1}{w(x + iy)} = u(x, y) + iv(x, y) \ .$$

Under the assumption that $P(z)$ has no complex zeros, $f(x + iy)$ is continuous and rational on any rectangle in the Cartesian plane. One may write

$$u = \frac{\sigma}{\sigma^2 + \tau^2}, \quad v = \frac{-\tau}{\sigma^2 + \tau^2} \ ,$$

where $w(x+iy) = \sigma(x, y) + i\tau(x, y)$. We noted earlier that f has continuous partial derivatives of all orders, and satisfies the Cauchy-Riemann equation $f_y(x + iy) = if_x(x + iy)$ everywhere. Furthermore, from $f(x + i0) = u(x, 0)$ is strictly positive, according to the first paragraph of this Section.

We will be working, as already suggested, with iterated integrals over a rectangular closed domain. No particular discussion of integrals or measures in several dimensions need enter in, only (iterated) Riemann or Jordan integrals of one variable.

Now we consider the closed *square* domain $\mathcal{D}_R = \{x, y \in \mathbb{R} | -R \leq x \leq R, 0 \leq y \leq 2R\}$. We start out with interest in the iterated integral expression

$$T_R = \int_0^{2R} \int_{-R}^{R} v_x(x, y)\, dx dy \ .$$

See Figure A.

Using Point 5 (FTC in definite integral form) we may simplify to

$$T_R = \int_0^{2R} [v(R, y) - v(-R, y)]\, dy \ . \tag{6}$$

On the other hand, if we replace v_x by $-u_y$ according to the C-R conditions for $f(x + iy)$, there results after reversing the integration by Proposition 1,

$$T_R = \int_{-R}^{R} [u(x, 0) - u(x, 2R)]\, dx \ . \tag{7}$$

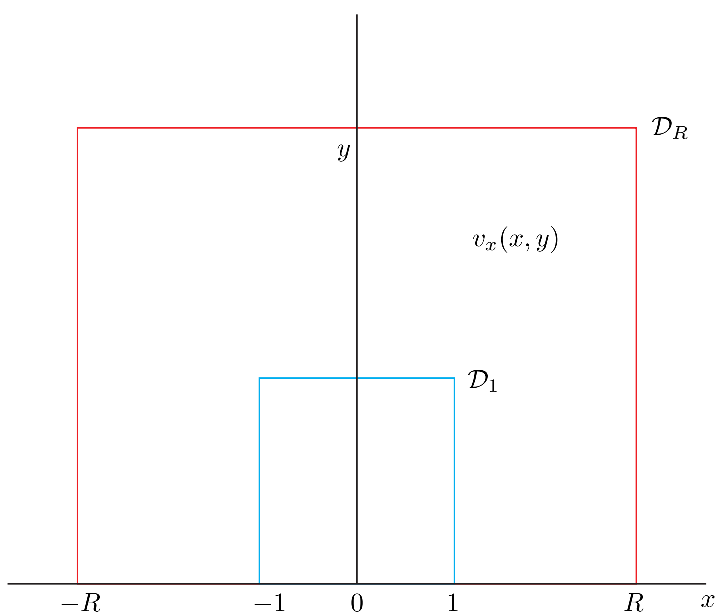

Figure A

Combining (6) and (7) leads to

$$\begin{aligned} \Lambda_R := \int_{-R}^{R} u(x, 0) &= \int_{-R}^{R} u(x, 2R) dx + \int_0^{2R} v(R, y) dy - \int_0^{2R} v\,(-R, y)\, dy \\ &= A_R + B_R + C_R \ , \end{aligned} \tag{†}$$

defining four integral quantities.

Recalling that $n = \operatorname{degr} P(z)$ was stipulated as even, so $u(x,0) = f(x+0i) > 0$ for all $x \in \mathbb{R}$, we must have $\Lambda_{R'} > \Lambda_R > 0$, whenever $R' > R > 0$. In particular, $\Lambda_1 > 0$.

By the "growth inequality" above, we are able to find $R_B > 1$ so that $|z| \geq R_B$ implies that $\frac{1}{2}|z^n| \leq |w(z)|$, hence

$$|f(x+iy)| \leq \frac{2}{|(x+iy)^n|} \ ,$$

and noting that $|v(R,y)| \leq |f(R+iy)|$, we obtain

$$|B_R| \leq \int_0^{2R} |v(R,y)| dy \leq 2 \int_0^{2R} \left| z^{-n} \right| dy \leq \frac{4R}{R^n} \leq \frac{4}{R}$$

whenever $R > R_B$; we used $n \geq 2$. But also $S_B > R_B$ can be chosen so that $R > S_B$ implies:

$$\frac{4}{R} \leq \frac{\Lambda_1}{4} \ ,$$

so we arrive at

$$|B_R| = \left| \int_0^{2R} v(R,y) dy \right| \leq \frac{\Lambda_1}{4}$$

whenever $R > S_B > R_B > 1$. Similarly for the term A_R, there exists $S_A > 1$ such that $R > S_A$ implies

$$|A_R| = \left| \int_{-R}^{R} u(x, 2R) dx \right| \leq \frac{\Lambda_1}{4} \ ,$$

and similarly a number $S_C > 1$ yielding

$$|C_R| \leq \frac{\Lambda_1}{4} \ ,$$

for $R > S_C$.

Taking $S = \max\{S_A, S_B, S_C\}$, we conclude from the preceding three inequalities that

$$(\ddagger) \qquad |A_R + B_R + C_R| \leq |A_R| + |B_R| + |C_R| \leq \frac{3}{4}\Lambda_1 < \Lambda_1 < \Lambda_R \ ,$$

whenever $R > S > 1$. But Formula (†) means that $A_R + B_R + C_R = \Lambda_R$, so we have arrived at a contradiction: the rational function

$$f(x+iy) = \frac{1}{w(x+iy)}$$

cannot be well-defined everywhere on $\mathbb{R}^2$, so also $P(z)$ must have a root $z_0 \in \mathbb{C}$. ∎

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