ABOUT CURVATURE AND LEVEL CROSSINGS

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I wonder what is known about the following simple situations from calculus. The first question involves involves geometric invariants of the graph of a function (of one variable) that tell you something about the number of zero- (or maybe also level-) crossings. The second related question asks what bounds there are on the total absolute curvature of a plane curve that is constrained to contain two given points, and possess given slopes at these points. The same question could be asked, instead of for plane curvature, for other measures of "curvyiness".

So suppose that h is a periodic function with period 1, of class C^2, at least twice continuously differentiable on $\mathbb{R}$, with zero mean,

$$\int_0^1 h(x)\,dx \quad .$$

I am interested in "measures" on h that give information on its *number* of zero-crossings in the interval [0, 1]. In case $h(x) = cos\, 2\pi n x$, we consider the form

$$\Lambda = \frac{\int_0^1 h(x)h''(x)\,dx}{\int_0^1 h^2(x)\,dx} \quad ,$$

and, motivated by Rice's formula, we obtain the number of zero-crossings $2n = \frac{1}{\pi}\sqrt{-\Lambda}$.

On the other hand, if h is the function whose graph consists of alternate upper and lower semi-circles of radius r, where $2k \cdot r = 1$, we will have k zeros, so we can use

$$\kappa = \int_0^1 \frac{|h''(x)|}{\{1 + h'(x)^2\}^{\frac{3}{2}}}\,dx \qquad \text{(total absolute plane curvature)} \quad .$$

Here we ignored the fact that h' is not defined everywhere. Another possible expression to look at is

$$\tilde{\kappa} = \int_0^1 \frac{h''(x)}{\{1 + h'(x)^2\}^{\frac{3}{2}}} \cdot \frac{h(x)}{|h(x)|}\,dx \quad .$$

How close an approximation is either of these to N_z, the number of zero-crossings, in general? Is there a way to bound

$$\int_0^1 \mathcal{F}_1 < N_z < \int_0^1 \mathcal{F}_2 \quad ,$$

for certain related forms $\mathcal{F}_i$?

Consider two points p, q in the plane with a direction vector at each one, v_p, v_q. Consider all curves

$$\chi : [-\epsilon, 1+\epsilon] \rightarrow \mathbb{R}^2 \quad ,$$

such that $\chi'(0) = v_p, \chi'(1) = v_q$.

Surely there is a minimum value for $\kappa(\chi)$, (the total absolute curvature of the plane curve χ that is in general greater than 0.

For example, taking $\chi(x)$ to be $cos\pi x$, and considering the interval $[0, \frac{1}{2}]$, we calculate

$$\kappa(\chi) = 1.9751$$

whereas for the cubic "spline" $\phi(x) = 32x^3 - 24x^2 + 1$ which has the same value and slope at $x = 0,\ .5$, we obtain

$$\kappa(\phi) = 1.9728$$

.

Both of these questions seem very classical in nature. A cursory look at Pólya and Szegö, "*Problems and Methods in Analysis*" did not turn up any related material.

Thanks very much for any suggestions you could make!

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Best wishes,
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Prof. Smolyanov O.G. (Oleg Georgievich) has agreed to receive printed matter
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Best wishes,
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