‘HARMONIC’ APPROACH TO FTA OF RAYMOND REDHEFFER

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Let $f(x)$ be a real polynomial with $n \geq 2$, $a_j \in \mathbb{R}$, $j = 0, \dots, n-1$, with $a_0 \neq 0$

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \ ,$$

We ask the question whether the *corresponding* complex polynomial (having real coefficients) possesses a root $z_0 = x_0 + iy_0$.

Here $f(z) = z^n + \sum_{j=1}^{n} a_{n-j} z^{n-j}$. A formal expansion of each sub-term $(x+iy)^n$ yields

$$g(x,y) := f(z) = u(x,y) + iv(x,y) \ ,$$

where u and v are real functions, in fact are harmonic conjugates satisfying the Cauchy-Riemann conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{at each } (x,y) \in \mathbb{R}^2$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{at each } (x,y) \in \mathbb{R}^2 \quad .$$

The C-R conditions can be proven for each polynomial term $\gamma(x+iy)^k$, hence for an arbitrary polynomial $g(x,y) = f(x+iy)$. Therefore setting

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \ ,$$

we obtain $\Delta u = \Delta v = 0$ everywhere: both $u(x,y)$ and $v(x,y)$ are harmonic functions.

Supposing that $f(z)$ might be a conterexample to FTA, no root $z_0 = x_0 + iy_0$ would exist, and $h(z) = \dfrac{1}{f(z)}$ will be well-defined and continuous for all $z \in \mathbb{C}$. Also, from the hypothesis we deduce that $u(x,y)^2 + v(x,y)^2 \neq 0$ for all $x, y \in \mathbb{R}$.

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Expanding $h(z)$ by $z = x+iy$ we derive by direct and tedious calculation (motivated by consideration of holomorphic functions) that

$$w(x,y) = \frac{u(x,y)}{u^2+v^2} \text{ is harmonic on } \mathbb{R}^2 ,$$

forming a pair of harmonic conjugates together with

$$l(x,y) = \frac{-v(x,y)}{u^2+v^2} .$$

We appeal briefly to elementary estimates on complex polynomials. Defining $C = \sum_{j=0}^{n-1} |a_j|$ one can derive

$$|f(z)| > \frac{1}{2}|z|^n \quad \text{for } |z| > 1 + C ,$$

[Hille, p. 208], using only the geometric series expansion and triangle inequality. Thus

$$|h(z)| < \frac{2}{|z|^n} \quad \text{for } |z| > R = 1 + C ,$$

hence also

$$|w(x,y)| < \frac{2}{|z|^n} \quad \text{since } |h(z)| = |w(x,y) + l(x,y)| .$$

Since $R \geq 1$, and $n \geq 2$, we can simplify further to obtain

$$|w(x,y)| = \frac{|u(x,y)|}{u^2+v^2} < \frac{2}{r} ,$$

where $r^2 = x^2 + y^2$ and $r > R$.

We have established that the harmonic function $w(x,y)$, defined everywhere, goes to zero in modulus as $x^2 + y^2$ goes to infinity.

Note that the solutions $\{(x_0, y_0)\}$ to the system

$$u(x,y) = 0$$
$$v(x,y) = 0$$

in fact yield all solutions $z_0 = x_0 \pm iy_0$. For since $f(z)$ has real coefficients, $(x_0, -y_0)$ will also be a valid solution. In case $y_0 = 0$, $f(x)$ is formally divisible by $(x - x_0)$, when $y_0 \neq 0$, $f(x)$ is divisible by $(x^2 - 2x_0x + x_0^2 + y_0^2)$.

Denote by D_α the closed disk $x^2 + y^2 \leq R^2$, so that by the Bolzano-Weierstrass Theorem, $w(x,y)$ assumes both an absolute maximum and minimum on D_α. We work with $w(x,y)$ in case $w(x_{\max}, y_{max}) \geq 0$, otherwise since $w(x_{\min}, y_{\min}) \leq 0$ we work with the harmonic function $-w(x,y)$, by renaming, whose maximum on D_α is non-negative.

Then let $M = w(x_{\max}, y_{\max})$ where $(x_{\max}, y_{\max}) \in D_\alpha$, but now pick $R' > R$ large enough so that $r \geq R'$ implies

$$(\dagger) \qquad \frac{-M}{11} \leq w(x,y) \leq \frac{M}{11} \quad .$$

Let D_β be the disk centered at $(0,0)$ of radius R'. Then certainly on the frontier ∂D_β the inequalities (†) hold and the point $(x_{\max}, y_{\max})$ interior to D_β gives $w(x_{\max}, y_{\max}) = M > M/11$, so there must exist an absolute interior maximum (x_*, y_*) for $w(x,y)$ in D_β, assuming the value

$$M_* = w(x_*, y_*), \quad (x_*, y_*) \in \operatorname{Int} D_\beta \ \text{ with } \ M \leq M_* \ .$$

On D_β define

$$q(x,y) = w(x,y) + \left(\frac{M_* - \frac{M}{11}}{2(R')^2}\right)(x - x_*)^2 \ .$$

There holds on the boundary ∂D_β

$$|q(x,y)| \leq |w(x,y)| + \frac{1}{2(R')^2}\left(M_* - \frac{M}{11}\right)(x - x_*)^2 \ ,$$

and also

$$|w(x,y)| \leq \frac{M}{11} \leq \frac{M_*}{11} \ ,$$

so we have holding on ∂D_β

$$|q(x,y)| \leq \frac{M_*}{11} + \frac{10}{22}M \leq \frac{12 M_*}{22} < M_*$$

since $(x - x_*)^2 \leq (R')^2$; note that $q(x_*, y_*) = M_* \geq M > 0$.

We observe that on D_β (or $B((0,0),R')$), the function $q(x,y)$ is of class C^2 and has an interior maximum somewhere, say

$$q(x_g, y_g) \geq M_* \ .$$

We compute the Laplacian

$$\Delta q(x,y) = \Delta w + \frac{\left(M_* - \frac{M}{11}\right)}{(R')^2} \geq \frac{10M}{11(R')^2} > 0 \quad .$$

Thus we have $\Delta q > 0$ everywhere on D_β including at the local internal maximum (x_g, y_g). Thus either

$$\frac{\partial^2 q}{\partial x^2} \quad \text{or} \quad \frac{\partial^2 q}{\partial y^2}$$

is strictly positive at (x_g, y_g), which is impossible according to the calculus of C^2 functions.

Considering $q(x, y)$ as the quotient of two polynomials, such an eventuality can also be ruled out by direct calculation.

The contradiction just arrived at shows that the "harmonic function" $h(x, y)$ cannot be justified, namely its denominator

$$u^2(x, y) + v^2(x, y)$$

must vanish somewhere in $\mathbb{R}^2$ indicating that $f(z)$ has a complex root. In other words, the original real-valued polynomial $f(x)$ possesses a factor of degree not greater than 2, in the variable x, as seen at the bottom of the second page.

References

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