THE BROUWER FIXED-POINT THEOREM

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ABSTRACT

We re-work two of the known proofs of the celebrated Brouwer theorem. The first is based on ideas of Milnor-Asimov originating in foliation theory and subsequently put into polished form by C.A. Rogers. The second is due to Y. Kannai and has been exposited by H. Flanders, M. do Carmo and others. Not to disregard the combinatorial proofs that also exist, based for instance on Sperner's Lemma, these two demonstrations seem definitive among those based on elementary (multivariate) calculus. Each has a direct plan of attack and uses only elementary tools in a compelling logical argument. Furthermore, the distinct approaches reinforce each other and strengthen our understanding of the BFPT and its implications.

THE FIXED-POINT THEOREM AND VOLUME UNDER A MAPPING

We use standard notation such as the inner product for vectors $x, y \in \mathbb{R}^n$, given by

$$x \cdot y = \langle x, y \rangle = \sum_{i=1}^{n} a_i \, b_i$$

where $x = \sum_{i=1}^{n} a_i \, e_i$, $y = \sum_{i=1}^{n} b_i \, e_i$ and e_i is the length n row vector with a single non-zero entry (namely, $1 \in \mathbb{R}$) in the i-th column. The norm is $\|x\| = (x \cdot x)^{1/2}$ and the closed unit hall B^n is defined by $B^n := \{x \in \mathbb{R}^n : \|x\| \leq 1\}$. Note that B^n is compact. The unit $n-1$ sphere S^{n-1} is the boundary ∂B^n, and the interior $B^n \backslash \partial B^n$ can be denoted $\mathrm{Int}(B^n)$ or $\dot{B}^n$.

Theorem. *(Brouwer 1909, Bohl 1904 for $n = 3$). Given a continuous mapping $f : B^n \to B^n$, the equation*

$$f(x) = x$$

always has a solution vector $x \in \mathbb{R}^n$ ("fixed point" for f).

We observe:

Proposition 1. *It is sufficient for the conclusion of the Theorem, that we prove it in the restricted case that $f(x)$ is a C^1 vector function (the Jacobian matrix varies continuously with $x \in B$).*

Proof. Given $f : B^n \to B^n$ merely continuous, note that

$$\delta = \inf\{\|x - f(x)\| = x \in B\}$$

can be assumed to be > 0. Otherwise, by the compactness of B^n, f will have a fixed point. But now f can be approximated by a C^1 function $g : B^n \to \mathbb{R}^n$ that moves the target vector by a distance at most $\delta/10$, so $\|g(x) - x\| \geq \delta/2$. The function g can be manufactured by convolving f with any reasonable smoothing kernel ("bump function") that is supported on B^n, or by appeal to the Weierstrass approximation theorem, resulting in an n-vector of n-variable polynomials. In either case consider the bigger closed ball $(1 + \delta/10)B^n$. There is a C^∞ radial shrinking map $\lambda : (1 + \delta/10)B^n \to B^n$ given by

$$\lambda(y) = \frac{10}{11} y\,,$$

and now

$$\tilde{g} = \lambda \cdot g$$

will map B^n to B^n, and each point will remain a distance at least $\delta/2 > 0$ from its target; hence $\tilde{g}$ is C^1 and has no fixed point. ■

Proposition 2. *Any C^1 mapping $g : B^n \to B^n$ has a fixed vector $x \in B^n$, so that $f(x) = x$.*

Proof. Together with Proposition 1 this Proposition proves the Theorem. The Proposition in turn follows from Propositions 3 and 4 below. The key to the demonstration is the Lipschitz property when applied to a C^1 vector function on a compact sub-space (such as B^n) of a metric space. This property follows from repeated application to the coordinate functions of the Mean Value Theorem from real analysis and can be found in any book a multivariate calculus, e.g. [C.H. Edwards]. In the article "Analytic Proofs...", [J. Milnor] proves this result concisely, namely that there exists a constant C such that for $x, y \in B^n$ there holds

$$\|g(x) - g(y)\| \leq C\|x - y\|.$$

The demonstration is made simpler by virtue of the convex nature of our (closed) domain B^n. ■

Proposition 3. *A fixed-point free C^1 mapping $g : B^n \to B^n$ gives use to a "retraction" mapping $f : B^n \to S^{n-1}$ such that $x \in S^{n-1}$ implies that $f(x) = x$. In other words, $f_{|S^n} = id_{S^n}$.*

Proof. One may see by coordinate geometry that if $y \in B^n$ and $g(y) \neq y$, there is a unique $z \in S^{n-1}$ such that $[g(y), y, z]$ represents points in order on an affine line in $\mathbb{R}^n$. For example, to express y as a weighted sum of $g(y)$ and z, the weighting coefficient is such that $0 \leq \tau \leq 1$. Then $f(y) = z$ is the definition of the new function, which is C^1 and satisfies the remaining claimed properties. ■

Hence we have reduced the principal Theorem by logic to the following

Proposition 4. *No C^1 retraction from $f : B^n \to S^{n-1}$ can exist.*

Proof. Suppose that f is C^1 with $f : B^k \to S^{n-1}$ and $f_{|S^{k-1}} = \mathrm{id}_{S^{k-1}}$.

Define $g(x) = f(x) - x$ and $f_t(x) = x + tg(x) = (1-t)x + tf(x)$, for all $\|x\| \leq 1$ and $0 \leq t \leq 1$. Using the Lipschitz property of the C^1 function $g(x)$, we obtain a constant C so that in the compact region B^n, we have

$$\|g(y) - g(x)\| \leq C\|y - x\|, \quad x, y \in B^n.$$

Supposing that $f_t(x) = f_t(y)$ we obtain

$$\|x - y\| = \|tg(y) - tg(x)\| \leq tC\|y - x\| .$$

In case $Ct < 1$ we must have $\|x-y\| = 0$ or $x = y$. Thus the mapping $f_t : B^n \to B^n$ is injective if the given t satisfies $0 \leq t \leq 1/C$.

For t small enough we wish to show that $f_t = B^n \to B^n$ is onto; here again

$$\begin{aligned} f_t(x) &= (1-t)x + tf(x) = x + tg(x) \\ g(x) &= f(x) - x \ . \end{aligned}$$

We seek for given $u_0 \in B^n$, a unique solution to $f_t(x) = u_0$. Now consider

$$h_t(x) = u_0 - tg(x) = u_0 - t\left(f(x) - x\right) = u_0 + x - f_t(x) .$$

Compute

$$|h_t(x) - h_t(y)| = t\,|g(y) - g(x)| \leq tC\|y - x\|$$

using the Lipschitz constant for $g(x)$. Therefore, if we make sure that $t < 1/C$, the function $h_t(x)$ must possess a fixed point on the complete metric space B^n, hence for all $u_0 \in B^n$, $f_t(x) = u_0$ has a (unique) solution $x_0 \in B^n$ by the fundamental Banach fixed-point theorem, see [C.H. Edwards]. If we choose $t_0 = \frac{1}{2C}$, we then have that for $0 \leq t \leq t_0$, the mapping $f_t : B^n \to B^n$ is both surjective and injective (one-to-one).

Next we come to a volume computation for $f_t(B^n)$. This volume is evaluated as the integral of a Jacobian determinant, namely

$$F(t) = \int_{B^n} \det\left(Jf_t(x)\right) dx = \int_{B^n} \left|\frac{\partial f_t}{\partial x}\right| dx = \int_{B^n} \det\left(I_n + tg'(x)\right) dx ,$$

where $g'(x)$ is the *matrix* $\left[\dfrac{\partial g_i}{\partial x_k}\right]$ and I_n is the $n \times n$ identity matrix.

The determinant under the integral above is thus a polynomial in t of degree at most n, whose coefficients are functions on B^n.

Put another way,

$$\left|\frac{\partial f_t}{\partial x}\right| = 1 + a_1(x)t + \cdots + a_n(x)t^n$$

so that we have $\operatorname{vol} f_t(B^n) = F(t) = b_0 + b_1 t + \cdots + b_n t^n$ when, taking $a_0 \equiv 1$,

$$b_j = \int_{B^n} a_j(x)dx \quad \text{for } j = 1, \ldots, k\,.$$

We note that for t in the interval $[0, t_0]$ we have

$$\operatorname{vol} f_t(B^n) = \operatorname{vol} f_0(B^n) = \operatorname{vol}(B^n) \neq 0\,.$$

But any polynomial $F(t)$ equals its McLaurin series which must in this case be constant, since $F(t)$ is constant on $[0, t_0]$. Hence also $F(1) = \operatorname{vol} f_1(B^n) > 0$.

However $f_1(B^n) \subset S^{n-1}$ so this latter volume must equal 0, resulting in a contradiction. Hence no such C^1 retraction $f : B^n \to S^{n-1}$ is possible.

We repeat this argument more concretely, from the fact that $\langle f_1(x), f_1(x)\rangle = 1$ for all x, which leads by Leibnitz' rule to

$$\left\langle \frac{\partial f_1}{\partial x_j}, f_1(x) \right\rangle = 0$$

identically for any $1 \le j \le n$, so that for example

$$\sum_{i=1}^{n} \frac{\partial f_1^i}{\partial x_j} \cdot f_1^i(x) = 0 \quad \text{for all } x \in B^n.$$

Thus a weighted sum of the columns of the Jacobian matrix

$$Jf_1 = \left[\frac{\partial f_1^i}{\partial x_j}\right]$$

is identically zero, as is $\det Jf_1$ and

$$\operatorname{vol}(f_1(B^n)) = F(1) = \int_{B^n} \det\left(Jf_1(x)\right) dx = 0\,.$$

This confirms the contradiction with $F(1) = \operatorname{vol}(B^n) \neq 0$ (see the following Section) as previously deduced from the hypothesis of the retraction f, finally proving Proposition 4. Hence, as noted, there follow also Proposition 2, Proposition 1 and the Theorem. ■

No-Retraction Using a Differential Form

We have seen how a C^1 Brouwer's FPT and indeed Brouwer's FPT in the continuous case, follows from Proposition 4, the "no-retraction theorem" for C^1 mappings. We offer an alternative proof of this, advocated by several authors, see [H. Flanders] and [A. Carbery]. The generalization to compact embedded n-manifolds seems first to be mentioned in [Y. Kannai], also Lima–do Carmo and M.W. Hirsch [Hirsch] have expanded on various approaches.

Let us take as our given data a *bounded* open set $T \subset \mathbb{R}^n$ whose boundary

$W = \partial T$ is smooth, that is it forms an embedded C^1 submanifold of dimension $n-1$. Our previous specific case of B^n and $S^{n-1} = \partial B^n$ certainly is covered by this description.

To approach full generality, we note a somewhat technical result.

Proposition 5. *If a continuous retraction $g : T \to W$ exists, that is where $t \in W$ implies $g(t) = t$, then also such a C^1 retraction (or even C^∞) exists.*

Proof. The proof based on the Weierstrass approximation theorem can be constructed as a generalization of Lemma 1 in [C. A. Rogers]. ■

Then to show that no such retraction can exist at all, we have only to prove the following.

Proposition 6. *There is no C^1 retraction $g = T \to \partial T = W$.*

Proof. Consider the class of C^1 mappings $\mathfrak{G} = \{\phi : T \to T \mid \phi_{|W} = \mathrm{id}_{|W}\}$. Note that $g \in \mathfrak{G}$.

Define the "definite integral" value

$$\Lambda = \int_T dg_1 \wedge dg_2 \wedge \cdots \wedge dg_n = \int_T \left| \frac{\partial g_i}{\partial x_j} \right| dx_1 \wedge \cdots \wedge dx_n \,,$$

where we work with Cartesian coordinates and observe the Jacobian determinant of g.

Note that the meaning of Λ amounts to the n-dimensional volume or measure of the smooth boundary W, which must equal zero.

More explicitly, since a tangent space Y_t is defined at each $t \in W$, the Jacobian matrix $\left|\frac{\partial g}{\partial x}\right|$ must have deficient rank ($< n$) at every $t \in T$. Hence the determinant is always $= 0$ and so is the integral Λ. (Also see the proof of Proposition 4).

On the other hand, Stokes' theorem implies that

$$\Lambda = \int_W g_1 dg_2 \wedge \cdots \wedge dg_n \,.$$

Choosing another smooth map $h = T \to T$, which leaves each boundary point fixed, that is, $h \in \mathfrak{G}$, we observe that $h_1(t) = g_1(t)$ for $t \in W$, in fact $h(t) = g(t)$.

Hence

$$\Lambda = \int_W h_1 dg_2 \wedge \cdots \wedge dg_n.$$

We may then apply Stokes' theorem "in reverse" obtaining

$$\Lambda = (-1)^{n+1} \int_T dg_2 \wedge \cdots \wedge dg_n \wedge dh_1 \,.$$

We repeat this process a total of n times giving a total parity of $+1$, so

$$\Lambda = \int_T dh_1 \wedge \cdots \wedge dh_n \,,$$

but we can select h to be the identity on T, $h(t) = t$, $(t \in T)$. In that case $\Lambda = \text{vol}(T) \neq 0$ since the measure of an open set in $\mathbb{R}^n$ is > 0. In the case of $T = B^n, n \geq 1$, we have

$$\text{vol}(T) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} > 0\,.$$

This volume formula is well-known and can be established in various ways, for instance by computing the expected volume of an n-ball whose radius is an exponentially distributed random variable with mean 1 [Personal communication, Fuchang Gao].

For completeness, the volume calculation, without using probability distributions, starts with the well-known definite integral

$$\int_{\mathbb{R}^n} e^{-\pi<x,x>} dx = 1\,.$$

We set Ω_n as the $(n-1)$-dimensional volume of $S^{n-1} = \partial B^n$. Scaling the volume formulas according to a radius variable, it is known that the volume of the sphere is the differential of that of the corresponding ball. From these considerations results $\text{vol}(B^n) = \frac{1}{n}\Omega_n$. The calculation is finished by evaluating the above integral using polar coordinates:

$$1 = \int_{\mathbb{R}^n} e^{-\pi<x,x>} dx = \int_{S^{n-1}} \left(\int_0^\infty e^{-\pi r^2} r^{n-1} dr \right) d\sigma$$
$$= \Omega_n \cdot \int_0^\infty e^{-\pi r^2} r^{n-1} dr = \frac{\Omega_n \Gamma(n/2)}{2\pi^{n/2}}\,.$$

The previous contradiction regarding the value of Λ implies that no C^1 retraction $g : T \to \partial T$ can exist in general, and in the particular case $T = B^n$ we may infer another demonstration of Brouwer's Fixed Point Theorem.

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