SELF-ADJOINT FILTERS ON A SPACE OF PERIODIC SIGNALS

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1. Introduction

We start with an inner-product space of functions $X = \{h : [0,1) \to \mathbb{C}^N\}$ or $X = \{h : [0,1) \to \mathbb{R}^N\}$. Then X can be considered as a space of functions of period 1 defined on the real numbers, with values in a complex or real Euclidean space.

We say that $R : X \to X$ is *shift invariant* (or simply *invariant*), when for $g(t) = h(t-a)$, the following holds,

$$(Rg)(t) = (Rh)(t-a),$$

for all $t \in \mathbb{R}$. Considering the components of such an invariant linear operator,

$$(Rg)(t) = (R^1 g(t), \dots, R^j g(t), \dots)^T,$$

they can be given as a vector of convolution operators, acting via "dot product" on the components of g:

$$R^j = (f_j^1 *, \dots, f_j^N *),$$
$$R^j(g_1(t), \dots, g_N(t)) = \sum_{i=1}^{N} f_j^i * g_i.$$

Here, the quantity f_j^i can be a "generalized function", or periodic distributional generalization of a function from $L_2(0,1)$ (see [6 Chap. 5]). With respect to the inner product on $L_2(0,1)^N$ given by

$$\langle g, h \rangle = \sum_{i=1}^{N} \int_0^1 g_i(\tau) \overline{h_i(\tau)}\, d\tau,$$

(or over $\mathbb{R}$ without the conjugation), the self-adjointness criterion for R, namely $\langle Rg, h \rangle = \langle g, Rh \rangle$, translates into the "Hermitian matrix" condition

$$f_i^j(t) = \overline{f_j^i(-t)}.$$

We may consider any one-dimensional signal $g \in L_2(0,1)$ as composed of a "DC" (direct current) term $g_{DC} = A \cdot \chi_{[0,1)}$, where $A = \int_0^1 g(t)\,dt$ and χ is the characteristic function of the interval, and an "oscillatory term" $g_O = g - g_{DC}$.

Let $Y \subset X$ be the inner product space of purely oscillatory signals (i.e., with vanishing mean value). We may restrict R to Y and project, obtaining (to overload the language), a linear mapping $R : Y \to Y$. Thus R is given by the composition

$$Y \xrightarrow{\iota} X \xrightarrow{R} X \xrightarrow{\pi} X/\mathbb{C} = Y.$$

Our main purpose is to show how our invariant, self-adjoint operator $R : X \to X$ induces a linear mapping $R : Y \to Y$ which has another specific kind of self-adjointness property

We will have to restrict the space X for the moment to the set of periodic functions $\mathcal{C}$ which possess left and right derivatives at every point of $[0,1)$. Such a function can be expanded in the countable (Fourier) basis $\{\mathbf{e}_k\}$,

$$\mathbf{e}_k = \exp[2\pi \imath k t], \quad k = 0, \pm 1, \pm 2, \dots,$$

as $g \sim \sum_{-\infty}^{+\infty} c_k \mathbf{e}_k$, that is as a discrete sum converging to $g(t)$ in the L_2 norm. If $N > 1$ then we can define $\mathcal{C}_N$ as a vector $(g_1(t), \dots, g_N(t))$ where each $g_i \in \mathcal{C}$.

We next define the "building blocks" into which g is decomposed according to continuous sums. For $a, b \in [0, \frac{1}{2}]$ we define

$$\Lambda_a(t) = \begin{cases} 1, & -a \le t \le a \\ 0, & a \le t \le -a \pmod 1 \end{cases}$$
$$D_b(t) = \delta(t + b - \frac{1}{2}) \qquad \text{(Dirac delta).}$$

Here Λ_q and D_b represent "momentum" and "position" variables respectively. Note that Λ is an even function (considered as periodic defined on $\mathbb{R}$). We next write (uniquely within our class of functions $\mathcal{C}$), $g_O(t) = g_{ev} + g_+$, where g_{ev} is even, and $g_+(t) = 0$ for $\frac{1}{2} < t < 1$.

The functions $\Lambda(t)$ and $D(t)$ form a generating set for $\mathcal{C}$, where we allow continuous as well as discrete linear combinations to form a "composite" function g. Thus

$$\text{(A)} \qquad \begin{aligned} g_{ev} &= \int_{a=0}^{\frac{1}{2}} \alpha_a \cdot \Lambda_a(t) \\ g_+ &= \int_{\tau=0}^{\frac{1}{2}} \beta_\tau \cdot D_\tau(t). \end{aligned}$$

We may write for the "coordinates" of a function in this non-orthogonal expansion,

$$\langle \Lambda_a | g = \alpha_a$$
$$\langle D_b | g = \beta_b$$

and can study the "matrix" of the operator R with respect to this "continuous basis".

Example. Let

$$g(t) = \begin{cases} 2, & 0 \le t < \frac{1}{2} \\ 1, & \frac{1}{2} \le t < \frac{3}{4} \\ 5t - \frac{7}{4}, & \frac{3}{4} \le t < 1 \end{cases}$$

. (See Figure 1.) In this case we have

$$\alpha_a = \begin{cases} 5\,dt, & 0 \le a < \frac{1}{4} \\ 1 + \frac{5}{2}\,dt, & a = \frac{1}{4} \\ 0, & \frac{1}{4} \le a \le \frac{1}{2} \end{cases}$$

and

$$\beta_b = \begin{cases} 1\,dt, & 0 \le b \le \frac{1}{4} \\ \frac{1}{2}\,dt, & b = \frac{1}{4} \\ (\frac{5}{4} - 5\,b)\,dt, & \frac{1}{4} < b \le \frac{1}{2} \end{cases}$$

.

The coefficients α_a and β_b are computed from (A) above. See also the formulas following (B) below.

Taking into account the invariant operator R we could now derive (The Three Formulas):

(1) $\langle \Lambda_a | R | \Lambda_b \rangle = \overline{\langle D_b | R | D_a \rangle}$, where $R|D_a\rangle$ means $R \circ D_a(t)$,
(2) $\langle \Lambda_a | R | D_b \rangle = -\overline{\langle \Lambda_b | R | D_a \rangle}$,
(3) $\langle D_a | R | \Lambda_b \rangle$ is purely imaginary and $\langle D_a | R | \Lambda_b \rangle = \langle D_b | R | \Lambda_a \rangle$.

These relations lead to the "self-adjointness" property of R acting on Y (the space of zero-mean signals). Let us define a symplectic form ("Poisson bracket") by means of

$$\{\Lambda_a, \Lambda_b\} = 0,$$
$$\{D_a, D_b\} = 0,$$
$$\{\Lambda_a, D_b\} = \jmath\,\delta_{a,b},$$

extended by additive bi-linearity and satisfying

$$\{x, y\} = \overline{\{y, x\}}, \quad \{\alpha x, \beta y\} = \alpha\overline{\beta}\{x, y\},$$

for $\alpha, \beta \in \mathbb{C}$.

Then we have that if $S = \jmath\, R, \quad \jmath = \imath = \sqrt{-1}$, the Three Formulas above imply that

$$\{Sx, y\} = \{x, Sy\}. \tag{*}$$

That is, $S = \jmath\, R$ has the self-adjointness property with respect to the form $\{,\}$, based on the "position" and "momentum" variables D_a, Λ_b (compare [1 Chap. 3]).

2. Case of Scalar-Valued Signals

We move toward demonstrating the Three Formulas of the Introduction in the case $N = 1.$. If $g, g' \in Y$ are expanded as in (A), we compute the Poisson bracket as a mixed (discrete and continuous) sum

$$\{g, g'\} = \int_{a=0}^{\frac{1}{2}} (\alpha_a \bar{\beta}'_a \, - \, \bar{\alpha}'_a \beta_a).$$

It is worth remarking that Λ_a is actually not purely oscillatory, so is not in Y. Also D_a has a DC component and even worse, is not a function so cannot lie in $X = \mathcal{C}$. The first difficulty can be resolved by taking as modified basis

$$\hat{\Lambda}_a = \Lambda_a - 2a\chi_{[0,1)}$$
$$\hat{D}_a = D_a - \chi_{[0,1)}.$$

Whether we take the original basis or the modified basis, the expansion coefficients α_a and β_b are easily seen to be the same.

The second difficulty is dealt with as our somewhat restrictive choice of operator R (those which are "convolution with a Hermitian function from $\mathcal{C}$") ensures that any "observable" applied to D_a yields a *bona fide* function.

We turn to verification of the Three Formulas of the Introduction. For Formula (1), we must examine $\langle \Lambda_a | R | \Lambda_b \rangle$ and $\langle D_a | R | D_b \rangle$, where $R = f*$ and f has the Hermitian property $f(-t) = \overline{f(t)}$.

First of all,

$$\text{(B)} \quad R|D_b\rangle = \int_0^1 f(\tau) D_b(t - \tau)\, d\tau = \int_0^1 f(\tau)\delta(t - \tau + b - \frac{1}{2})\, d\tau = f(t + b - \frac{1}{2}).$$

If $f \in \mathcal{C}$, we have

$$\langle \Lambda_b | f = \frac{1}{2}\{df(-b^+) + df(-b^-)\} + f(-b^+) - f(-b^-),$$

where $df(-b^+) = \lim_{t \to -b^+} \frac{\partial f}{\partial t}(t) \cdot dt$ and so forth.

Now $\langle D_b | f$ is obtained from

$$\begin{aligned} \langle D_b | f_+ &= \frac{1}{2}\{f_+(\frac{1}{2} - b^+) + f_+(\frac{1}{2} - b^-)\} \cdot dt \\ &= \{f(\frac{1}{2} - b^+) + f(\frac{1}{2} - b^-)\} \cdot dt - \frac{1}{2}\{f(\frac{1}{2} + b^-) + f(\frac{1}{2} + b^+)\} \cdot dt. \end{aligned}$$

Hence

$$\text{(C)} \qquad \langle D_a | R | D_b \rangle = \frac{1}{2} \left\{ \begin{array}{l} f(-a + b^+) - f(a + b^-) + \\ f(-a + b^-) - f(a + b^+) \end{array} \right\} \cdot dt\,.$$

To examine $\langle \Lambda_a | f * | \Lambda_b \rangle$, we obtain

$$f * |\Lambda_b\rangle = \int_{t-b}^{t+b} f(\tau)\, d\tau = h(t).$$

Since $h(t)$ is continuous, there is no finite part in $\langle\Lambda_a|h$ and there results

$$\begin{aligned}\langle\Lambda_a|f*|\Lambda_b\rangle &= \frac{1}{2}\left(\frac{dh(t)}{dt}\bigg|_{-a^+} + \frac{dh(t)}{dt}\bigg|_{-a^-}\right)\cdot dt \\ &= \frac{1}{2}\{f(-a+b)^+ - f(-a-b)^+ + f(-a+b)^- - f(-a-b)^-\}.\end{aligned}$$

Using $f(-t) = \overline{f(t)}$, we get

$$\langle\Lambda_a|f*|\Lambda_b\rangle = \frac{1}{2}\left\{\begin{array}{l}\overline{f(a-b)^-} - \overline{f(a+b)^-} + \\ \overline{f(a-b)^+} - \overline{f(a+b)^+}\end{array}\right\}.$$

Now switching a and b in (C) we obtain

$$\langle D_b|R|D_a\rangle = \frac{1}{2}\left\{\begin{array}{l}f(a-b)^+ - f(a+b)^- + \\ f(a-b)^- - f(a+b)^+\end{array}\right\},$$

proving Formula (1).

Note for future reference

$$f(a)^+ = \lim_{t\to a^+} f(t) = \lim_{t\to a^+} \overline{f(-t)} = \lim_{t\to -a^-} \overline{f(t)} = \overline{f(-a)^-},$$

and

$$f'(a)^+ = \lim_{t\to a^+} f'(t) = \lim_{t\to a^+} -\overline{f'(-t)} = \lim_{t\to -a^-} -\overline{f'(t)} = -\overline{f'(-a)^-}.$$

So we have

$$\text{(D)}\qquad \begin{aligned} f(a)^+ &= \overline{f(-a)^-} \\ df(a)^+ &= -\overline{df(-a)^-}.\end{aligned}$$

Next we establish Formula (2), starting with $\langle\Lambda_a|f*|D_b\rangle = \langle\Lambda_a|g(t)$ where $g(t) = f(t - \frac{1}{2} + b)$. Expanding as in (B) gives

$$\begin{aligned}\langle\Lambda_a|f*|D_b\rangle &= g(-a)^+ - g(-a)^- + \frac{1}{2}\{dg(-a)^+ + dg(-a)^-\} \\ &= f(-a+b-\frac{1}{2})^+ - f(-a+b-\frac{1}{2})^- \\ &\quad + \frac{1}{2}\{f'(-a+b-\frac{1}{2})^+ + f'(-a+b-\frac{1}{2})^-\}\,dt.\end{aligned}$$

Interchanging a and b gives

$$\langle\Lambda_b|f*|D_a\rangle = f(a-b-\frac{1}{2})^+ - f(a-b-\frac{1}{2})^- + \frac{1}{2}\{f'(a-b-\frac{1}{2})^+ + f'(a-b-\frac{1}{2})^-\}\,dt.$$

Replacing $f(t)$ by $\overline{f(-t)}$ throughout the right-hand expression and making use of relations (D) gives

$$\overline{f(b-a-\frac{1}{2})^-} - \overline{f(b-a-\frac{1}{2})^+} - \frac{1}{2}\{\overline{f'(b-a-\frac{1}{2})^-} + \overline{f'(b-a-\frac{1}{2})^+}\}\,dt.$$

But these terms add up to $-\overline{\langle\Lambda_a|f*|D_b\rangle}$, as was to be shown to establish Formula (2).

Next we must consider $\langle D_a|f*|\Lambda_b\rangle$. We obtain $f*|\Lambda_b\rangle = \int_{t-b}^{t+b} f(\tau)d\tau = h_b(t)$. Letting

$$w_{a,b} \stackrel{\text{def}}{=} h_b(\frac{1}{2}-a) - h_b(\frac{1}{2}+a),$$

we then have $\langle D_a|h_b = w_{a,b}\cdot dt$. Right- and left-hand limits are not needed here since $h_b(t)$ is continuous for all $0 < b < \frac{1}{2}$, being the convolution of ordinary piece-wise continuous functions. In fact,

$$w_{a,b} = \left\{\int_{\frac{1}{2}-a-b}^{\frac{1}{2}-a+b} - \int_{\frac{1}{2}+a-b}^{\frac{1}{2}+a+b}\right\} f(\tau)\,d\tau.$$

In the second integral we set $\sigma = -\tau$ and get

$$\begin{aligned}-\int_{\sigma=\frac{1}{2}-a+b}^{\frac{1}{2}-a-b} f(-\sigma)\,d(-\sigma) &= -\int_{\frac{1}{2}-a-b}^{\frac{1}{2}-a+b} f(-\sigma)\,d\sigma \\ &= -\int_{\frac{1}{2}-a-b}^{\frac{1}{2}-a+b} \overline{f(\sigma)}\,d\sigma.\end{aligned}$$

Hence

$$\langle D_a|f*|\Lambda_b\rangle = 2\jmath\cdot dt\cdot\Im\int_{\frac{1}{2}-a-b}^{\frac{1}{2}-a+b} f(\tau)\,d\tau.$$

This quantity is pure imaginary. Now consider

$$w_{b,a} = \left\{\int_{\frac{1}{2}-a-b}^{\frac{1}{2}+a-b} - \int_{\frac{1}{2}-a+b}^{\frac{1}{2}+a+b}\right\} f(\tau)\,d\tau \quad .$$

Depending on the configuration of the limits of integration on the t-axis, we can decompose the two integrals. In a typical case, $b < a,\ a+b < \frac{1}{2}$, we obtain

$$w_{b,a} = \left\{\int_{\frac{1}{2}-a-b}^{\frac{1}{2}-a+b} + \int_{\frac{1}{2}-a+b}^{\frac{1}{2}+a-b} - \int_{\frac{1}{2}-a+b}^{\frac{1}{2}+a-b} - \int_{\frac{1}{2}+a-b}^{\frac{1}{2}+a+b}\right\} f(\tau)\,d\tau.$$

The two middle integral terms cancel exactly and we retain only the leftmost and rightmost terms which add to $w_{a,b}$. All other configuration cases are handled similarly, so we have shown that $w_{b,a} = w_{a,b}$ which is equivalent to Formula (3). In other words we have

$$\langle D_a|f*|\Lambda_b\rangle = -\overline{\langle D_b|f*|\Lambda_a\rangle},$$

since $w_{a,b}$ is pure imaginary. We have verified the Three Formulas in the scalar-valued case.

3. Vector-Valued Signals: The Main Result

The space of signals $X = L_2(0,1)^N$ with values in $\mathbb{R}^N$ or $\mathbb{C}^N$ is isometric with $L_2(0,1)$ but the explicit description of X makes it possible to examine a weaker invariance property than that considered in the last section. In other words, the class of allowed "invariant" signals is larger. As in the Introduction we have

$$[R\vec{h}_\tau](t) = [R\vec{h}](t-\tau),$$

where $\vec{h}_\tau(t) = [h_1(t-\tau), \dots, h_N(t-\tau)]^T$.

Combining invariance with the Hermitian property $\langle R\vec{h}, \vec{g}\,\rangle = \langle \vec{h}, R\vec{g}\rangle$ for the inner product on $L_2(0,1)^N$ gives a characterization of such an operator as a vector of convolutions with generalized functions

$$[R\vec{g}](t) = \left[R^1 g(t), \dots, R^j g(t), \dots\right]^T.$$

The Hermitian condition becomes

$$f_i^j(t) = \overline{f_j^i(-t)}.$$

We continue the discussion from the Introduction and seek to decompose $X = X_{dc} + Y$ where Y comprises signals whose components $y_i, i = 1, \dots, N$ all have zero mean. This is to be done in such a way that all the operators R that we are considering are put into a special form. Exactly as in Section 2, we decompose each scalar component $g_i(t)$ of $\vec{g} \in L_2(0,1)^N$ into a continuous sum of $\{\Lambda_a^i, D_a^i\}$. Defining the Poisson brackets, sometimes viewed as giving "Heisenberg relations" (compare [9, Form. 3.11]) as

$$\begin{aligned}\{\Lambda_a^i, \Lambda_b^j\} &= 0,\\ \{D_a^i, D_b^j\} &= 0,\\ \{\Lambda_a^i, D_b^j\} &= \jmath\, \delta_{a,b}\delta_{ij},\end{aligned}$$

and setting $S = \jmath R$, we obtain for the vector case:

Theorem I.

$$(\Omega) \qquad \{Sx, y\} = \{x, Sy\},$$

which is a self-adjointness property for S.

This equation (Ω) is equivalent to the following, which has an explicit "Hermitian" meaning. Consider the special functions we have defined to be variables, and set $p_a^i = \Lambda_a^i$, $q_a^i = D_a^i$ and define $\Phi = \jmath RJ$, where J is the symplectic operator generated by

$$\begin{aligned}p_a^i &\mapsto q_a^i\\ q_a^i &\mapsto -p_a^i.\end{aligned}$$

Then the equivalent formulation of (Ω) is

Corollary 1.

(E) $$\langle \vec{g}\,|\Phi|\vec{h}\,\rangle = \overline{\langle \vec{h}|\Phi|\vec{g}\,\rangle}.$$

Proof. The complete proof makes use of the identities of Section 2 and will appear elsewhere in a longer article.

Analogous to the Three Formulas of the Introduction are three formulas in the vector case (VC):

(1) $\langle \Lambda^i_a|R|\Lambda^j_b\rangle = \overline{\langle D^j_b|R|D^i_a\rangle}$,
(2) $\langle \Lambda^i_a|R|D^j_b\rangle = -\overline{\langle \Lambda^j_b|R|D^i_a\rangle}$,
(3) $\langle D^i_a|R|\Lambda^j_b\rangle = -\overline{\langle D^j_b|R|\Lambda^i_a\rangle}$.

Using Formula (3.VC) for example, we obtain one case of (E), namely that

$$\begin{aligned}
&\langle q^j_b|\Phi|q^i_a\rangle = \langle q^j_b|\jmath RJ|q^i_a\rangle = \langle q_b|\jmath R| - p^i_a\rangle \\
&= -\jmath\langle q^j_b|R|p^i_a\rangle = \jmath\overline{\langle q^i_a|R|p^j_b\rangle} = -\overline{\langle q^i_a|\jmath R|p^j_b\rangle} \\
&= \overline{\langle q^i_a|\jmath RJ|q^j_b\rangle} = \overline{\langle q^i_a|\Phi|q^j_b\rangle}.
\end{aligned}$$

Other cases of (E) are similar.

4. Real Operators in Fourier Space

If we restrict the space X to consist of functions with real values, the invariant operators $\{R\}$ under consideration become real symmetric operators. As long as R can be represented as $R = [f^j_i *]$ where $\{f^j_i\}$ belong to $\mathcal{C}$, R is a compact operator. We derive a result on the eigen-spectrum of R, first using a Fourier approach. We then sketch a proof of the same result using our decomposition of $\mathcal{C}$. This approach, if it can be made rigorous, may be more generally applicable than the Fourier method.

Let us introduce a little Fourier notation. Consider still that R takes complex values. Then, examining the range (image) space by coordinates, there is a Fourier generating set as in the Introduction, consisting of

$$\mathbf{e}^r_k = \exp[2\pi\imath kt], \quad k = 0, \pm 1, \pm 2, \dots,$$

living in the r-th copy of $L_2(0,1)$. Then let E_k be generated by the $\mathbf{e}^r_k$, $r = 1, \dots N$. Due to the invariance of R, the action of R preserves wave number k, and we get a linear mapping $A_k : E_k \to E_k$. The mapping $A = \oplus A_k : E \to E$, where

$$E = \dots E_{-1} \perp E_0 \perp E_1 \dots \quad \text{orthogonal direct sum},$$

can be considered as the Fourier transform of R, as shown in [4 Sec. 8.3].

If $P_0 : E \to E_0$ is the natural projection, we may consider $P(x)$ as the DC component of the signal x, and $\hat{P}(x)$ its oscillatory component, where $\hat{P} : E \to \hat{E}$ is the projection and $\hat{E} = \oplus_{1\le|k|} E_k$.

Thus A becomes an infinite direct sum of $N \times N$ matrices A^k.

Theorem II. *If R is a real convolution operator (invariant) where the matrix $[f_i^j(t)]$ is symmetric, then any eigenvalue λ of R on the oscillatory space Y rules an even-dimensional eigenspace. Explicitly, the eigenvectors of R occur in pairs of the form*

$$\vec{m}_1(t) = [\alpha_1 \sin 2\pi kt, \ldots, \alpha_N \sin 2\pi kt]$$
$$\vec{m}_2(t) = [\alpha_1 \cos 2\pi kt, \ldots, \alpha_N \cos 2\pi kt].$$

Proof. Since R is real, compact and symmetric, its eigenvalues λ are real and have finite-dimensional eigenspaces on X as in [1]. The matrix A^k can be written $[\hat{f}_{ij}^k]$, and direct calculation shows that A^k is Hermitian (not necessarily real), from the property $f_j^i(t) = \overline{f_i^j(-t)}$.

Using the reality of f_i^j, we compute

$$\hat{f}_{ij}^{-n} = \int_0^1 f_j^i(t)\, e^{2\pi\imath(-nt)}\, dt = \int_0^1 \overline{f_j^i(t)}\, \overline{e^{2\pi\imath nt}}\, \overline{dt}$$
$$= \overline{\int_0^1 f_i^j(t) e^{2\pi\imath nt}\, dt} = \overline{\hat{f}_{ij}^n}.$$

Therefore A^n and A^{-n} are mutually conjugate (and Hermitian) and $A^n \cdot w = \lambda w$ implies that $A^{-n} \cdot \bar{w} = \lambda \bar{w}$. Thus both $\frac{w+\bar{w}}{2}$ and $\frac{w-\bar{w}}{2\imath}$ are real eigenvectors for λ. Recalling that the basis elements in each of the N coordinates were $\mathbf{e}_n = \exp(2\pi\imath nt)$ gives the conclusion of the Theorem.

We briefly discuss another approach to this result, based on Theorem I. We work with the p, q inner product given by $[p_a^i, p_b^j] = \delta_{i,j}\, \delta_{ab}$ and so on. Since we take RJ to be real, it is anti-symmetric on Y by Theorem I. Set $U = -J$ and $T = -JRJ$. We seek a subspace $W \subset Y$ with three properties,

(1) $T(W) \subset W$ (invariance),
(2) any finite dimensional subspace of W which equals a direct sum of T-invariant subspaces, also equals an *orthogonal* direct sum of T-invariant subspaces,
(3) $W \cap U(W) = \{0\}$.

The union of a (totally ordered) chain of such subspaces $\{W_i\}$ also satisfies the conditions, and is an upper bound for the chain. By Zorn's Lemma [8, Chap. 7], there is then a *maximal* subspace W that satisfies these conditions. Then one may show, setting $V = U(W)$, that W and V are metrically closed and that

$$Y = W \oplus V \quad \text{orthogonal.}$$

Furthermore

$$[JRw, w'] = 0$$
$$[Jv, v'] = 0,$$

where $w, w' \in W,\ v, v' \in V$. This shows that V is invariant under the action of R since JRv is orthogonal to V and must be in W, and Jw is orthogonal to W hence must be in V. Since $V = JW$, the action of R on V is just an orthogonal basis change from its action on W and this action has the same eigenvalues and eigenvectors related by J. Since R is symmetric we expect the eigenvalues of $R|_W$ and $R|_V$ to have distinct eigenspaces in Y (non-derogatory).

It would be even better if we could arrange a change of basis so that

$$R \simeq R_1 \oplus R_2$$

where the structure of the factors R_i is nearly identical. In a finite-dimensional framework, this result is true (see [7]). The "sub-direct sum" that we did establish serves to indicate that the eigenspaces of R on Y have an even dimension, as stated in Theorem II. Of course since our collection of generators $\{p_a^i,\, q_a^i\}$ is a basis in the "continuous" sense, this second (non-Fourier) version of the proof will require more elaboration.

5. Finite Periodic Signals

If $f(t)$ is defined only for

$$t = 0, \pm\frac{1}{2K+1}, \ldots, \pm\frac{K}{2K+1},$$

then the subject under consideration is actually matrix theory, for which see [3] or [5]. Suppose that we have $N \times N$ complex matrices $A_0, A_{\pm 1}, \ldots, A_{\pm K}$, which are *Hermitian*. Thus we may regard A_i as

$$A_i : V^i \to V^i \quad \text{where}$$

V^i has a basis $\mathbf{e}_j^i$, $j = 1, \ldots, N$.

Now $A = \oplus A_i$ operates naturally on $V = \oplus V^i$, but we may write A differently depending on the chosen basis of V. Take $V \simeq U \simeq S^0 \oplus \hat{S} \oplus \hat{T}$ with preferred bases

$$\begin{aligned} S^0 &= \{\mathbf{s}_1^0, \ldots, \mathbf{s}_N^0\} = V^0, \\ \hat{S} &= \{\mathbf{s}_1^1, \ldots, \mathbf{s}_1^K, \mathbf{s}_2^1, \ldots, \mathbf{s}_N^K\}, \\ \hat{T} &= \{\mathbf{t}_1^1, \ldots, \mathbf{t}_1^K, \mathbf{t}_2^1, \ldots, \mathbf{t}_N^K\}. \end{aligned}$$

Since there is an inclusion of $\hat{U} \stackrel{\text{def}}{=} \hat{S} \oplus \hat{T}$ in U, the value of Ax for $x \in \hat{U}$ is defined in U, and by projection (throwing away terms of S^0), also in $\hat{U}$.

Thus $\hat{A} : \hat{U} \to \hat{U}$ is defined. The dimension of $\hat{U}$ is $2K \cdot N$, so we can set

$$J_{KN} = \begin{bmatrix} \mathbf{0} & -I_{KN} \\ I_{KN} & \mathbf{0} \end{bmatrix}.$$

If a subspace U_j is spanned by $\{\mathbf{e}_j^i\}, i = \pm 0, \ldots, \pm K$, then another basis $\{\mathbf{r}_j^i\}$ is defined by

$$\mathbf{r}_j^i = \frac{1}{\gamma} \cdot \sum_{-K \le q \le K} \omega^{-iq} \mathbf{e}_j^q,$$

where $\omega = \exp(\frac{2\pi\imath}{2K+1})$ and $\gamma = \sqrt{2K+1}$ in $\mathbb{C}$.

Then let

$$
\begin{aligned}
\mathbf{s}_j^0 &= \sum_{0\le|q|\le K} \mathbf{r}_j^q = \gamma \mathbf{e}_j^0 \\
\mathbf{s}_j^1 &= \mathbf{r}_j^1 + \mathbf{r}_j^{-1} \\
\mathbf{s}_j^2 &= \mathbf{s}_j^1 + \mathbf{r}_j^2 + \mathbf{r}_j^{-2} \\
&\vdots \\
\mathbf{s}_j^K &= \mathbf{s}_j^{K-1} + \mathbf{r}_j^K + \mathbf{r}_j^{-K} \\
\mathbf{t}_j^1 &= \mathbf{r}_j^{-K} \\
&\vdots \\
\mathbf{t}_j^K &= \mathbf{r}_j^{-1}
\end{aligned}
$$

be the new basis of U_j. The change of basis we have effected is in fact *unimodal* (that is, with $|\det| = 1$).

Then with respect to the new basis, $\jmath A \cdot J_{KN}$ is seen to be a Hermitian matrix. This result should be checked by hand or with an automated symbolics tool for $N = 2$, $K = 2$ say. Then the manipulations required for the general case should become evident.

6. Conclusion

We have shown how certain invariant operators (filters) of importance in physics and communication science, acting on a space of periodic signals, give rise to a Hermitian structure for the coefficients in a particular "continuous" basis of the signals, modulo constant signals. These particular filters are actualized as "convolution with a nice function" which rules out, e.g., translation operators. Further research could consider more general classes of operators and functions (distributions). Then the decomposition that we gave would be the "first-order" part of a more general decomposition. For example, we might need to see the coefficients of a basis element δ' in a sum of Dirac δ's. Such an extended decomposition should also show symmetry characteristics, akin to the symplectic symmetry on zero-mean signals that we showed. Quantum-like models, where the "observables" are taken from a class of 1- or N-dimensional filters, are of interest both in physics and in mathematics. Cognizance of any symmetry inherent in such models is critical to their application and further exploration.

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