SELF-ADJOINT FILTERS ON A SPACE OF PERIODIC SIGNALS

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Abstract

The foundation of much of modern physics rests on the theory of operators on a separable Hilbert space. Mainly the operators considered are bounded and Hermitian, or self-adjoint. In signals engineering and in optics, space- and time-invariant operators (filters) play a critical role. A particular model of quantum theory that requires the "observables" to have both self-adjointness and invariance properties could be useful. We introduce identities that will be important in the study of such a model.

Let G be a Hermitian, invariant operator on $X = L_2(0,1)^N$, the Hilbert space of periodic signals with values in $\mathbb{R}^N$ or $\mathbb{C}^N$. In the paper we work with a subspace of L_2 to avoid technicalities, namely $\mathcal{C}$, the set of functions in $L_2(0,1)$ possessing left and right derivatives at all $t \in [0,1)$. When $N = 1$, G may be realized by a left convolution $g*$, where g is a (generalized) function satisfying

$$g(\tau) = \overline{g(-\tau)}, \quad \text{arguments defined} \pmod 1 .$$

Let X_{dc} denote the "direct current" part of the space of signals X, and Y a suitable complement in X of X_{dc} (consisting of "zero-mean" signals for example). The intention is to decompose Y into a basis $\{p_a, q_b\}$, of "position" and "momentum" variables (with real parameters), so that all such operators G are simultaneously put into a form that is "symplectic" with respect to this (non-orthogonal) basis.

That is, if J is the symplectic operator (generalizing symplectic matrix) that takes p_a to q_a and q_a to $-p_a$, then

$$\Phi = \jmath G\, J$$

($\jmath = \sqrt{-1}$) has the Hermitian property

$$\langle x|\Phi|y\rangle \quad = \quad \overline{\langle y|\Phi|x\rangle}$$

A similar simultaneous decomposition holds for $N > 1$ (linear symmetric filters on signals in N-space), though this is not obvious.

Finally, if the functions considered have *real* values, each eigenvalue of a symmetric invariant operator on the "oscillatory" space Y presides over an even-dimensional eigenspace. This may be shown by a Fourier argument or alternatively using our continuous basis $\{p_a, q_b\}$. To extend all of these results to a larger convolution algebra $\mathcal{B} \supset \mathcal{C}$ would be of interest.

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