COMPLEX ODD-DIMENSIONAL ENDOMORPHISM AND TOPOLOGICAL DEGREE

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Synopsis

H. Derksen has produced an interesting proof of the Fundamental Theorem of Algebra that relies on properties of certain operators on linear spaces of symmetric matrices over the complex numbers. The "analytic" part of his proof seems to lie in the well-known fact that an odd-order polynomial over the *reals* $\mathbb{R}$ does have a root. Using the companion matrix construction, Derksen poses the problem as finding an eigenvector for any finite endomorphism $T : \mathbb{C}^n \to \mathbb{C}^n$. A proof by induction proceeds according to the size of the integer ν where $n = 2^\nu(2m+1)$, just as in the proof by P.S. Laplace (1795), and later proofs due to Gauss, Gordan and E. Artin! A reference for the history of the FTA is [Fine & Rosenberger].

Remarkably, the base case of Derksen's induction, $\nu = 0$, uses a different matrix representation than does the inductive step. It could be convenient to have a separate proof of this 0-level result, one that is consistent with the "linear algebra" framework. Simply put, we need to exhibit an eigenvector for T when n is odd.

Toward this goal, one works with parametrized linear algebra over the real projective plane $\mathbb{R}P^2$ (and sometimes over the projective line). The degree of the original polynomial is revealed in the rank of a pair of Whitney-summed hyperplane bundles: the trivial $2n$-bundle, and a $2n$-fold sum of copies of the canonical ("Hopf") line bundle. Given some operator T_0 on $\mathbb{C}^n$ with no eigenvalue-eigenvector, results from the 1980s and 90s, on the topology of matrix spaces, show that these must be B-isomorphic bundles, with base space $B = \mathbb{R}P^2$. This improbable situation can be repudiated by the application of the (total) Stiefel-Whitney class. However, our intention in this article is to operate within a limited range of mathematics. This range certainly includes the definitions and basic facts of the homotopy concept, including its application to line bundle constructions over a compact Hausdorff space.

We keep the reader informed of alternate proofs involving various methodologies. But in principle we avoid homology groups, cohomology rings and their operations, characteristic classes, K-theories with computation by spectral sequence, classifying spaces (and their cellular architecture!), as well as techniques from differential topology, such as critical points, transversality, and "pre-image" manifolds. Simplicial approximation, measure and density results are held to a minimum.

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What does enter in are propositions pertaining to Brouwer degree for dimension two only. We refer to the "classical" Borsuk-Ulam theorem which is an introductory result of geometric topology. How to express the topological degree (for maps on $\mathbb{S}^2$) is handled according to preference, either through linear approximation, differential forms such as "curvatura integra", or even by counting pre-image points.

We hope that the one or two new technical observations made here will be applicable outside the limited domain where we have used them. Whether or not some truly new results might arise, it is worthwhile to seek the underlying reason as to why an "odd transformation" of a complex space has an "axis", as does a spinning spheroid in mundane 3-space.

References

B. Fine and G. Rosenberger, *The Fundamental Theorem of Algebra*, UTM Springer-Verlag, New York, 1997.