# A DERIVATION OF BEDROSIAN'S FORMULA

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20 May 1999

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## Introduction

We give a derivation of a formula conjectured by the electrical engineer S.D. Bedrosian, for all the Fibonacci numbers, which is illustrative of several principles of linear algebra and analysis, and a suitable classroom example. There are other proofs, but the one presented here does not involve complex numbers (so roots of unity do not appear), nor do the Fundamental Theorem of Algebra, resultants, discriminants, or the Binet formula come in. The basic facts that are used are

i) the elementary expansion properties of the determinant;
ii) addition laws of the trigonometric functions;
iii) the fact that the determinant of a square matrix whose eigenvectors span its domain equals the product of the matrix eigenvalues.

In fact, in point iii) it is enough to know that "determinant equals product of eigenvalues" when the eigenvalues are distinct. Alternatively, one could use the fact that the diagonalizable (or "non-derogatory") property holds when the matrix in question is symmetric. It is interesting to observe the *trade-off* of concepts, and their level of depth, that occurs in different proofs of this Formula. This particular proof may motivate a generalization involving special functions, other than sin and cos. Relating algebraic properties of the function to to an integer matrix, might yield some new transcendental identity. That is, the transcendental quantities of interest would obey a recursion similar to the one that defines the Fibonacci sequence.

## Bedrosian's Formula

For $n = 1, 2, \ldots$, let the usual Fibonacci numbers be given by $f_1 = 1, f_2 = 1, f_3 = 2, f_4 = 3, f_5 = 5$ and so forth based on the recursive formula

(2) $$f_{n+2} = f_{n+1} + f_n \qquad n > 1.$$

The identity in question is given in [1] or [2] as

(3) $$f_n = \prod_{k=1}^{r} (3 + 2\cos\frac{2\pi k}{n})$$

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where $r = \lfloor \frac{n-1}{2} \rfloor$, using the "greatest integer" function. When $n = 1$, the product in (3) is empty, which is interpreted as 1. The formula (3) was noticed in the context of certain problems of electrical circuitry, and was proved in certain cases in [1].

To prove (3), we take $\theta = \dfrac{2\pi}{n}$, and form the set of $r$-vectors $\mathcal{V}_r = \{v_k\}$, $\quad k = 1, \ldots, r$ by

$$v_k = \begin{bmatrix} \sin k\theta \\ \sin 2k\theta \\ \vdots \\ \sin rk\theta \end{bmatrix}.$$

Let $\mathcal{A}_r, \mathcal{B}_r, \mathcal{C}_r$ be $r \times r$ matrices defined by

$$[\mathcal{B}_r]_{ij} = 1 \text{ if } |i - j| = 1, \quad = 0 \text{ otherwise},$$
$$[\mathcal{C}_r]_{ij} = -1 \text{ if } i = j = r, \quad = 0 \text{ otherwise},$$

and

$$\mathcal{A}_r = \mathcal{B}_r + \mathcal{C}_r.$$

Thus in particular we have

$$\mathcal{A}_4 = \begin{bmatrix} 0 & 1 & & \\ 1 & 0 & 1 & \\ & 1 & 0 & 1 \\ & & 1 & -1 \end{bmatrix}, \quad \mathcal{B}_4 = \begin{bmatrix} 0 & 1 & & \\ 1 & 0 & 1 & \\ & 1 & 0 & 1 \\ & & 1 & 0 \end{bmatrix}.$$

Note that the vector $v_k$ actually depends on $n$ as well as $k$ for its definition. Next we wish to establish the following eigenvalue formulas:

$$\mathcal{A}_r v_k = 2\cos k\theta \cdot v_k \text{ for } n = 2r + 1\ , \quad \mathcal{B}_r v_k = 2\cos k\theta \cdot v_k \text{ for } n = 2r\ . \tag{4}$$

These vector identities follow from considering the entries row by row. For the first row, the result is the same as the identity $\sin 2k\theta = 2\cos k\theta \sin k\theta$. For all rows $f$ with $1 < f < r$, identities (4) are equivalent to the formula (valid for all angles $\theta$),

$$\sin k(f-1)\theta + \sin k(f+1)\theta = 2\cos k\theta \sin kf\theta,$$

which is proved by applying the addition law $\sin(\mathbf{a} + \mathbf{b}) = \sin \mathbf{a} \cos \mathbf{b} + \sin \mathbf{b} \cos \mathbf{a}$ to both terms on the left-hand side. Then we should establish the equality of the last entries on both sides of (4). For the $\mathcal{A}_r v_k$ last row calculation, we need to see that

$$\theta = \frac{2\pi}{2r+1} \text{ implies } \sin k(r-1)\theta - \sin kr\theta = 2\cos k\theta \sin kr\theta.$$

We quickly verify this trigonometric identity by proving that $\sin k(r+1)\theta + \sin kr\theta = 0$, which follows from $\sin(2k\pi - \mathbf{a}) = -\sin \mathbf{a}$, where $\mathbf{a}$ is taken as $kr\theta = \dfrac{kr\,2\pi}{2r+1}$. (The angles $kr\theta$ and $k(r+1)\theta$ add up to a multiple of $2\pi$.) Finally, we need the last row calculation for $\mathcal{B}_r v_k$. This amounts to $\sin k(r-1)\theta = 2\cos k\theta\, \sin kr\theta$ which is equivalent to $\sin k(r+1)\theta = 0$. This last identity follows immediately from the fact that $\theta = \dfrac{2\pi}{2r+2}$.

Thus we have established formulas (4), so for both the case $n$ is odd and $n$ is even, the numbers $2\cos k\theta$ $k = 1, \dots, r$ constitute a set of eigenvalues for $\mathcal{A}_r$ and $\mathcal{B}_r$ respectively. These values are all distinct, since the function $\cos(t)$ is strictly decreasing on the domain $(0, \pi)$. Thus in particular, the set of vectors $\mathcal{V}_r$ is linearly independent, and the matrices $\mathcal{A}_r$ and $\mathcal{B}_r$ are non-derogatory (their eigenvectors form a basis of the domain $\mathbb{R}^d$).

If $I$ is the $r \times r$ identity matrix

$$I = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \\ & & \ddots & \\ 0 & 0 & & 1 \end{bmatrix}_{r\times r},$$

then let

$$A_r = \mathcal{A}_r + 3I\,, \quad B_r = \mathcal{B}_r + 3I.$$

Using the same eigenvectors as before, it is clear that

$$\{3 + 2cos\, k\theta\}, \quad k = 1, \dots, r$$

constitutes a complete set of eigenvalues for $A_r$, and for $B_r$, in the odd and even cases respectively. Hence,

$$\begin{aligned} \det A_r &= \prod_{k=1}^{r}(3 + 2\cos\frac{2k\pi}{2r+1}) \\ \det B_r &= \prod_{k=1}^{r}(3 + 2\cos\frac{2k\pi}{2r+2}). \end{aligned} \tag{5}$$

But we have

$$A_r = \begin{bmatrix} 3 & 1 & & 0 & 0 \\ 1 & 3 & & 0 & \vdots \\ & & \ddots & & \\ & & & 3 & 1 \\ 0 & \dots & & 1 & 2 \end{bmatrix}_{r\times r}, \quad B_r = \begin{bmatrix} 3 & 1 & & 0 & 0 \\ 1 & 3 & & 0 & \vdots \\ & & \ddots & & \\ & & & 3 & 1 \\ 0 & \dots & & 1 & 3 \end{bmatrix}_{r\times r},$$

so that by the Laplace expansion, $\det B_r =$

$$\begin{vmatrix} 3 & 1 & & & \\ 1 & 3 & & & \\ & & \ddots & & \\ & & & 3 & 1 \\ & & & 1 & 2 \end{vmatrix}_{r\times r} + \begin{vmatrix} 3 & 1 & & & \\ 1 & 3 & & & \vdots \\ & & \ddots & 1 & \\ & & 1 & 3 & 0 \\ & & & 0 & 1 \end{vmatrix}_{r\times r} =$$

$\det A_r + \det B_{r-1}$. Similarly we can obtain the recursive identity $\det A_r = 2 \cdot \det B_{r-1} + \det A_{r-2}$. Taking as a starting point $r = 1$, so that $\det A_1 = 2$ and $\det B_1 = 3$, these recursions immediately imply

$$\begin{aligned} f_{2r+1} &= \det A_r \\ f_{2r+2} &= \det B_r, \end{aligned}$$

which by the formulas (5) is equivalent to Bedrosian's formula.

## Appendix

We now give another proof of Bedrosian's formula, which does not use any concept or property of linear mapping, matrix, eigenvalue, or special function, but does employ the solvability of a polynomial over the complex numbers $\mathbb{C}$, root of unity, and the Binet formula. For an elementary discussion of the Golden Ratio and the formula of Binet for the Fibonacci sequence, one may wish to consult [3]. The Binet identity is usually *derived* analytically in courses on number theory using a generating function. To be sure, once this formula is conjectured it is straightforward to *verify* it; see for example [5], page 123. The present elegant proof of the Bedrosian identity was found on the spot by a well-known but unnamed topologist upon hearing the problem stated. The incidental derivation of Bedrosian's formula which is done in [4], in the course of proving results in enumerative combinatorics, must contain the essence of his/her approach.

Consider the equation in one indeterminate $p(x) \equiv [c(x-c)]^n - 1 = 0$ where $c = \dfrac{1+\sqrt{5}}{2}$. All $n$ solutions come about from further solving $c(x-c) = \omega^k$, $k = 1, \ldots, n$ where $\omega$ is a primitive $n$-th root of unity. Thus the solutions are

$$x_k = \omega^k c^{-1} + c, \quad k = 1, \ldots, n.$$

Note that $c^{-1} = \dfrac{\sqrt{5}-1}{2}$ so in fact $c^{-1} + c = \sqrt{5}$ and $-c^{-1} + c = 1$. Now $\dfrac{p(x)}{c^n} \equiv q(x)$ is a monic polynomial (the coefficient of the monomial $x^n$ of highest degree equals 1), so the product of all the roots of $q(x)$ must equal $(-1)^n$ times the constant term of $q(x)$.

But the roots of $q(x)$ are the same as the roots of $p(x)$, and their product is

$$\Lambda = \prod_{k=1}^{n} x_k = \prod_{k=1}^{n} (\omega^k c^{-1} + c).$$

Making use of the facts noted above, that $x_n = \sqrt{5}$, and if $n$ is even, that $x_{\frac{n}{2}} = 1$, we may write as valid for $n > 0$ that

$$\Lambda = \sqrt{5} \prod_{k=1}^{r} (\omega^k c^{-1} + c)(\omega^{n-k} c^{-1} + c).$$

The $k$-th factor in the above product can we computed using the complex representation of roots of unity. In fact it equals

$$(e^{\imath k \frac{2\pi}{n}} c^{-1} + c)(e^{\imath(n-k)\frac{2\pi}{n}} c^{-1} + c) = c^2 + c^{-2} + 2\cos\frac{2\pi}{n} = 3 + 2\cos 2\pi n.$$

Thus $\Lambda$ is just $\sqrt{5}$ times the product expression in Bedrosian's formula.

There remains only to evaluate the constant term in $q(x)$ directly. It is in fact

$$(-1)^n c^n - \frac{1}{c^n} = (-1)^n \left(\frac{\sqrt{5}+1}{2}\right)^n - \left(\frac{\sqrt{5}-1}{2}\right)^n.$$

We have that $\Lambda$ equals $(-1)^n$ times this expression, so dividing both sides of this equality by $\sqrt{5}$ gives

$$\frac{\Lambda}{\sqrt{5}} = \prod_{k=1}^{r}(3 + 2\cos\frac{2\pi k}{n}) = \frac{1}{\sqrt{5}}\left[\left(\frac{\sqrt{5}+1}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] = f_n$$

according to the Formula of Binet [3].

## References

[1] S.D. Bedrosian, *The Fibonacci Numbers via Trigonometric Expressions*, J. of the Franklin Institute **295** (1973), 175–177.

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[3] H.E. Huntley, *The Divine Proportion*, Dover, New York, 1970.

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[5] M.D. Spivak, *The Joy of Tex*, American Mathematical Society, Providence, Rhode Island, 1990.

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