CYCLES AND SPANNING TREES

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Dedicated to the Memory of Marvin Phelps Epstein

INTRODUCTION

Given a finite, connected, (unoriented) graph G (or a *multigraph* where edges may have positive integral multiplicities), one may consider maximal trees that are present as subgraphs of G. If the edges of G are suitably labeled, by distinct letters or colors, such trees can similarly be distinguished from each other. The famous formula of Cayley asserts that if G is the complete graph on n vertices, then the number of such distinct trees is n^{n-2}.

The vertex-vertex incidence matrix M of G, defined as the diagonal vertex degree matrix minus the adjacency matrix, yields a method for enumerating the "spanning" (all vertices of G are included) tree subgraphs of G. One deletes the last, or any other, row and column of M, and takes the determinant of the resulting matrix $\tilde{M}$. The fact that this calculation enumerates the spanning trees follows from expressing M as the product of the vertex-edge incidence matrix B and its transpose using the Cauchy-Binet Theorem, [Sw&Th] or [Ko]. This method is based on the vertices and their adjacency. We argue that a formulation utilizing cycles is more natural. A set of cycles defines an intersection form, given roughly by counting the common edges of two cycles with sign.

Consider an edge e in G whose removal disconnects G. Then e belongs to any spanning tree of G. Hence if the ends of e are identified and e removed, the resulting multigraph G' has equally many spanning trees. But it is also clear that e cannot belong to any "geometric cycle" (precise definitions below). Continuing in this way we arrive at a multigraph H, with as many maximal trees as G which has no disconnecting edges, and each edge belongs to a cycle. Hence one might expect that a way to count spanning trees in terms of a basis of cycles could be found.

The Theorem that confirms this expectation is given in Section 1. It asserts that the number of labeled spanning trees is equal to the determinant of a cycle-cycle incidence matrix (intersection form). The set of cycles selected must form an "integral basis" of the algebraic cycles, but in particular, a basis set of geometric cycles will serve. Notice that is is not necessary in this case to delete any row and column.

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For certain graphs the cycle methodology requires less computation. Also, consider the case of the graph (one-skeleton) of a polyhedron. If the 2-dimensional boundary surface of the polyhedron is topologically a sphere, say that the graph is "spherical". A graph is spherical if and only if it is planar. Equivalently, the polyhedron should be realizable up to "piecewise-linear equivalence" as a *convex* polyhedron P in 3-space. Then P has a natural dual P^*, where faces go to vertices, edges go to (orthogonal) edges, and vertices go to faces.

Theorem 0. *The spanning trees of a convex polyhedron P are equinumerous with those of P^*.*

For instance, among the Platonic solids, the icosahedron and dodecahedron both contain $5,184,000$ distinct labeled maximal trees as subgraphs of their one-skeletons.

As further examples we consider the generalization to multigraphs of graphs studied by [Vo&Wa]. These multigraphs K have n vertices with the group of integers modulo n acting freely as a group or graph isomorphisms for K. Using the standard vertex method one obtains algorithms for the number of spanning trees, but these involve arithmetic with irrational numbers. Using the cycle-based method one can often obtain another algorithm where only integral polynomial calculations enter in.

In particular, the question of how to compute the Fibonacci numbers is relevant to this discussion. The defining algorithm

$$F_n = F_{n-1} + F_{n-2}$$

involves only integral calculation, but requires finding the entire sequence up to n to get F_n. The Binet formula, on the other hand, is "closed form" but in principle requires the calculation of an indefinite number of digits of $\sqrt{5}$.

In Section 2 a basis of cycles is constructed for connected n-rotational graphs. This basis seems to be natural with respect to the rotational property, and we apply it to several special cases, treated in Section 3. The first case gives a derivation of the formula for $\Omega(G_n)$, where G_n is the $(1,2)$ graph [Kl&Go], which uses only integer algebra. Kleitman and Golden's derivation involves embedding the graph into a continuum, and Vohra and Washington's method involves essential use of algebraic numbers. This comparison is carried out for the graph $G_n(1,3)$ as well. Here Vohra and Washington's method leads to a sequence of numbers that are integers since they are 1) algebraic integers, 2) invariant under a Galois action over $\mathbb{Q}$. By our dual approach, on the other hand, this sequence is seen to be "integrally computable" (recursively via integral polynomials). Some provisional definitions of these terms are set forth.

These examples hardly comprise a theory, but raise questions such as, if a sequence of integers is algebraically computable, under what conditions is it integrally computable? Certain integers arise in the calculation of $\Omega G_n(1,3)$ for which it is not clear that they are integrally computable.

Section 1. A Dual Construction for Maximal Tree Subgraphs

A *multigraph* G consists of a set V, called vertices, as set E, called edges, and a mapping g which assigns to each edge e one unordered pair of distinct vertices,

$$g(e) = \{v, w\}$$

. Here the vertices v, w are said to constitute the *ends* of the edge e, or collectively, the *border*. We say that v and w are *incident* to e, and *adjacent* to each other. A *path* in G consists of an alternating sequence of distinct vertices and distinct edges

$$p = (v_0, e_0, v_1, \ldots, v_{k-1}, e_{k-1}, v_k),$$

where $g(e_i) = \{v_i, v_{i+1}\}$.

Definition 1. A multigraph G is *connected* if for any two distinct vertices v, w there is a path as above with $v_0 = v$, $v_k = w$.

A *geometric cycle*, or *circuit* consists of a path together with an edge e_k whose ends are the initial and final vertex of the path. Hence $g(e_k) = \{v_0, v_k\}$. The definition differs slightly from that in [B-B], since we are dealing with the more general multigraphs.

Definition 2. An *orientation* for an edge e consists of a choice of one of its ends, called the *head* vertex $\mathrm{head}(e) \in g(e)$. An orientation for G is a mapping head : $V \to E$ giving an orientation on each vertex. The remaining vertex for a given orientation is called $\mathrm{tail}(e)$.

Let us endow G with a fixed orientation *head* once and for all. We will often write for an edge e, $e = (v, w)$, where $v = \mathrm{tail}(e)$, $w = \mathrm{head}(e)$. Then let $C_0(G)$ be the vector space (free $\mathbb{Q}-$module) generated by the set of vertices V, known as the *zero-chains*. Similarly let $C_1(G)$ be the *one-chains*, freely generated over $\mathbb{Q}$ by the set of edges E. The orientation leads to a homomorphism (linear mapping)

$$\partial : C_1(G) \to C_0(G)$$

given by $\partial(e) = \mathrm{head}(e) - \mathrm{tail}(e)$ on generators, and extended by linearity.

Let B be the $n \times m$ vertex-edge incidence matrix of a connected multigraph G. Here G has the n vertices $\{v_1, \ldots, v_n\}$, and m edges (each with multiplicity 1) $\{e_1, \ldots, e_m\}$. Then

$$B_{ij} = \begin{cases} 1, & \text{if } \mathrm{tail}(e_j) = v_i \\ -1, & \text{if } \mathrm{head}(e_j) = v_i \\ 0, & \text{otherwise.} \end{cases}$$

See p. 38 in [B-B].

If D is the $n \times n$ diagonal degree matrix, and A is the $n \times n$ vertex adjacency matrix defined by

$$A_{ij} = \text{multiplicity of the edge with border } \{v_i, v_j\},$$

if this edge exists, and 0 otherwise. In other words, A_{ij} gives the number of $1 \leq k \leq m$ such that $g(e_k) = \{v_i, v_j\}$. Furthermore, $D_{ii} = \sum_{j=1}^{n} A_{ij}$. Let superscript t denote matrix transpose.

Proposition 1. *The matrices BB^t and $D - A$ are equal.*

This is Theorem 6, p. 38, of [B-B]. Letting $\tilde{B}$ be the result of deleting a row (say the last) from B, we arrive at

$$\tilde{B}\tilde{B}^t = \tilde{M},$$

where $\tilde{M}$ was given in the Introduction.

Next choose a subset P of the n columns of $\tilde{B}$ and let $\tilde{B}_P$ be the square matrix obtained by restricting to these columns. Similarly, $\tilde{B}^t_P$ is the square matrix formed from the corresponding rows of $\tilde{B}^t$. Then the Cauchy-Binet theorem [Sw&Th] asserts that

$$\sum_P \det\tilde{B}_P \ \det\tilde{B}^t_P = \det\tilde{M}.$$

But it is well-known and not difficult to see that if the edges belonging to the columns P give rise to some cycle, the $\det\tilde{B}_P = 0$, and if not (in which case they form a spanning tree for G), one has $\det\tilde{B}_P = \pm 1$. hence $\det\tilde{M}$ gives precisely $\Omega(G)$, the number of spanning trees.

Now a geometric cycle gives a 1-chain via the fixed orientation. If the cycle c has edges $\epsilon_0, \ldots, \epsilon_l$ be the edges of c, where

$$\epsilon_0\xi_0\epsilon_1\xi_1 \ldots \epsilon_l\xi_l$$

(with ξ_j is the j-th vertex) is the natural ordering of the geometric cycle. Then define $\tilde{c} \in C_1(G)$ as $\tilde{c} = \sum_{k=0}^{l} f_k\epsilon_k$ where

$$f_k = \begin{cases} +1, & \text{if head}(\epsilon_k) = \xi_k \\ -1, & \text{if tail}(\epsilon_k) = \xi_k. \end{cases}$$

Then it is easy to see (or refer to a text on simplicial homology theory such as [Gr]) that

$$\partial\tilde{c} = 0.$$

A 1-chain σ satisfying $\partial\sigma = 0$ is called an *algebraic cycle*; these cycles form a submodule $Z(G) \subset C_1(G)$. One may show as in Theorem 5, p. 36, of [B-B], that the set of geometric cycles $\mathcal{C} = \{c\}$ generate the entire submodule $Z(G)$.

Next we adopt notation to allow consideration of **integral** chains and cycles. Write $C_0(G)_{\mathbb{Z}}$ for the module of algebraic 1-chains with integral coefficients, similarly $C_1(G)_{\mathbb{Z}}$, and $Z(G)_{\mathbb{Z}}$ (the *integral cycles*) to be $Z(G) \bigcap C_1(G)_{\mathbb{Z}}$.

Definition 3. Let $U \subset Z(G)_{\mathbb{Z}}$ be a finite subset. Then if any algebraic cycle $\sigma \in Z(G)_{\mathbb{Z}}$ can be written

$$\sigma = \sum g_u u \qquad \text{with } g_u \in \mathbb{Z},$$

we say that U is an *integral spanning set* for the integral cycles.

Definition 4. If U is an integral spanning set for the integral cycles of G and is $\mathbb{Z}$-linearly independent, we say that U is an *integral basis* for the integral cycles.

Observation. From Lemma 1 below it follows immediately that, for a connected multigraph, if a basis U for cycles consists of *geometric* cycles, it is an *integral* basis of $\mathbb{Z}$-cycles.

In particular, if the set U is linearly independent and consists of $m - n + 1$ elements (which are geometric cycles), then it is an integral basis.

Lemma 1. *Let W be a basis (in the vector space sense) over $\mathbb{Q}$ of the cycles $Z(G)$ for a connected multigraph G. Then if, writing an arbitrary element w_i of W in terms of edges*

$$w_i = \sum_j \lambda_{ij} e_j, \qquad \text{with } \lambda_{ij} = \pm 1 \text{ or } 0,$$

then each integral (algebraic) cycle z of G can be expressed as a linear combination

$$z = \sum \kappa_i w_i, \qquad \text{with } \kappa_i \in \mathbb{Z}.$$

That is, W is an integral basis of $\mathbb{Z}$-cycles.

Proof. Given $z = \sum \xi_i e_i$, $\xi_i \in \mathbb{Z}$, $e_i \in E(G)$, we have to show that z is an integral linear combination of the finite set $W = \{w_i\}$. If G has no integral cycles, hence is a tree, the conclusion is trivial. Suppose that for some w_f there is an edge e_ρ that occurs *only* in w_f. That is, $\lambda_{f\rho} = \pm 1$, but $\lambda_{g\rho} = 0$ for all $g \neq f$. Consider the multigraph $G_\rho = G - \{e_\rho\}$, which is still a connected multigraph since e_ρ occurs non-trivially in a cycle of G. The set of cycles $W_f = W - \{w_f\}$ is still $\mathbb{Q}$-linearly (and $\mathbb{Z}$-linearly) independent and has $(m - 1) - n + 1$ elements, and hence is a $\mathbb{Q}$-basis for the cycles of G_ρ.

Now write $z = \sum \gamma_i w_i$, $\gamma_i \in \mathbb{Q}$. If $\gamma_f = 0$, we may consider z to be a cycle in the multigraph G_ρ, so by an induction on the number of edges we obtain

$$z = \sum \kappa_i w_i, \qquad \kappa_i \in \mathbb{Z},$$

as was to be proved.

Alternatively if $\gamma_f \neq 0$, then $z' = z - \gamma_f w_f$ may be regarded as a cycle in G_ρ. From the edge description of z we get

$$(\xi_\rho - \gamma_f \lambda_{f\rho}) = 0.$$

Since $\xi_\rho \in \mathbb{Z}$ and $\lambda_{f\rho} = \pm 1$, we have $\gamma_f \in \mathbb{Z}$. But z' has by an induction hypothesis an integral expression in the basis W_f and hence $z = z' + \gamma_f w_f$ is integrally expressible over W as well, which is what had to be proved.

In general there may not be an edge e_ρ that occurs only in one cycle w_f. Suppose anyway that $\lambda_{f\rho} = \pm 1$, and suppose that another $w_k \in W$ contains e_ρ non-trivially, i.e., $\lambda_{k\rho} \neq 0$. Then let $\tilde{w}_k = w_k - \lambda_{k\rho}\lambda_{f\rho} w_f$. Then $\tilde{w}_k$ is an integral cycle that does not contain e_ρ non-trivially. Continuing in this way with all $w_k \in W$, $\quad k \neq f$ and replacing them with $\tilde{w}_k$ as necessary we obtain a new set of cycles $\tilde{W}$. Since

$\tilde{w}_k = w_k \pm w_f$, a cycle z is expressible by elements of W with integer coefficients if and only if it is integrally expressible by $\tilde{W}$, and $\tilde{W}$ is still a basis of $\mathbb{Q}$-cycles. But $\tilde{W}$ has w_f as an element and e_ρ satisfies the "exclusive" condition used earlier in the proof, for which case the Lemma was verified. Hence the Lemma holds in all cases.

Next suppose that U is given as an integral basis of integral cycles for a connected multigraph G. If $U = \{u_i\}, \quad i = 1, \ldots, r,\ r = m - n + 1$, we have

$$u_i = \sum_{j=1}^{m} f_{ij} e_j, \quad f_{ij} \in \mathbb{Z}.$$

Theorem 1. *Let $F = [f_{ij}]$ as above. Then if $R = FF^t$, we have $\Omega(G) = detR$.*

Proof. Recall the vertex-edge incidence matrix B. Selecting a set of $n-1$ edges S of G gives $n-1$ columns of B, and an $n-1 \times n-1$ submatrix $\tilde{B}_S$ of $\tilde{B}$, upon deleting a fixed vertex (row). If $T = E \setminus S$ is the complementary set of edges, one similarly obtains an $m-n+1 \times m-n+1$ submatrix of F by restricting to those columns, which is denoted F_T. We know that $\det \tilde{B}_S = \pm 1$ or 0 depending on whether the edges of S form a tree or do not.

We must show

1.1 $\left|\det \tilde{B}_S\right| = 1 \Rightarrow |\det F_T| = 1$.

1.2 $\det \tilde{B}_S = 0 \Rightarrow \det F_T = 0$.

For (1.1), suppose that $\det \tilde{B}_S = \pm 1$, and S gives a tree. Consider an edge $t \in T = E = S$. Then the subgraph whose edge set is $S \cup \{t\}$ has a unique algebraic cycle z_t such that $z_t \cdot t = +1$. Here we use the natural inner product

$$C_1(G) \times C_1(G) \to \mathbb{Q};$$

in other words, the t term of z_t has coefficient equal to 1. Furthermore, for any $t' \neq t$ in T, we also have $z_t \cdot t' = 0$. The uniqueness of this cycle follows since if there were two such cycles z_1 and z_2, both $z_1 - t$ and $z_2 - t$ could be realized as distinct paths from head(t) to tail(t). But in the tree S, only one such path can exist.

Thus, ranging over $t \in T$, there is a set of cycles $W = \{z_t\}$, $z_t \in Z(G)_{\mathbb{Z}}$. The set W is linearly independent, since if $\sum \lambda_t z_t = 0$, then $t_j \cdot \sum \lambda_t z_t = \lambda_j = 0$ for arbitrary j. By Theorem 5 of [B-B], W is also a spanning set for cycles over $\mathbb{Q}$. Hence by Lemma 1 above, W is an *integral* basis of cycles.

Now suppose that the given integral basis U is the same (as a set) as W. Then the matrix F_T has a single non-zero entry $+1$ in the row corresponding to $u_1 = z_{t_1}$. That entry is exactly in the column corresponding to t_1. Similarly, each row and column has precisely one non-zero entry equal to $+1$, so F_T is a permutation matrix with determinant equal to ± 1.

If the given basis U is different from W, then since both are integral bases of a $\mathbb{Z}$-module, there is an $r \times r$ unimodular integral matrix L such that

$$U = LW,$$

where $W = \begin{bmatrix} w_1 \\ \vdots \\ w_r \end{bmatrix}$, and $U = \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix}$ are treated as column vectors. In the construction of F_T, if we change the basis, we get

$$F(U)_T = LF(W)_T,$$

which since $\det L = \pm 1$, equals ± 1 as was to be proved. This finishes (1.1).

To establish (1.2), we suppose that $\det \tilde{B}_S = 0$, so the set of $n-1$ edges S contains a geometric cycle $\mathbf{c}$. Let $z_{\mathbf{c}}$ be the corresponding algebraic cycle. But for any $t \in T$, clearly

$$z_{\mathbf{c}} \cdot t = 0. \tag{1.3}$$

Since U is a basis, let

$$z_{\mathbf{c}} = \sum_i^r \eta_i u_i, \quad \eta_i \in \mathbb{Z}.$$

But

$$u_i = \sum_{t_j \in T} f_{ij} t_j \ + \sum_{s_j \in S} f_{ij} s_j .$$

Then by (1.3), $z_{\mathbf{c}} \cdot t_k = \sum \eta_i f_{ik} = 0$ for each $k \in T$. Since $z_{\mathbf{c}}$ was a non-trivial cycle, the row vector $\eta = [\eta_i]$ is not identically zero. Hence the matrix $[f_{ik}]$, $i = 1, \ldots, r$, $t_k \in T$ has a vanishing determinant. But this is exactly the matrix F_T, so we have proved (1.2).

The Theorem follows since clearly now

$$\tilde{B}_S \tilde{B}_S^t = F_T F_T^t$$

and $\det R = \sum_T \det F_T \cdot \det F_T^t$ by the Cauchy-Binet theorem, so we obtain

$$\Omega(G) = \det \tilde{M} = \sum_S \det \tilde{B}_S \det \tilde{B}_S^t .$$

This completes the proof of the Theorem.

The following discussion will culminate in a proof of Theorem 0. Let P be a convex polyhedron embedded in 3-space. Associated with P is a collection of faces $F = \{f_i\},$, $i = 1, \ldots, n_F$, edges $E = \{e_i\}$, $i = 1, \ldots, n_E$, and vertices $V = \{v_i\}, i = 1, \ldots, n_V$. Let P_1 be the graph consisting of the 1-skeleton of P with edge-vertex incidences induced from P. The *border* (geometric boundary) of a face f consists of a geometric cycle c_f of this graph P_1. Now since the 2-complex P is the geometric boundary of a compact 3-complex, it has a "natural" orientation given on faces by the right-hand rule (the thumb should point outward). An orientation in the graph sense may be chosen arbitrarily for P_1, but we now regard this orientation as fixed.

With respect to this orientation, the geometric cycle c_f gives rise to an algebraic cycle z_f in which each edge occurs with multiplicity $+1$, -1 or 0. The collection $\mathcal{C} = \{c_f\}$, where f ranges over all distinct faces, has the following property.

Claim. Given any proper subset $\mathcal{D} \subset \mathcal{C}$, there exists an edge $e \in E(P_1)$ such that there is exactly one geometric cycle $d \in \mathcal{D}$ that contains e non-trivially as an edge.

Proof sketch. This result was clear to the ancients. It is not necessary that P be convex for the Claim to hold, only *connected* as a simplicial complex. Now each edge e of $\mathcal{D}$ occurs at most twice (in two elements $d \in \mathcal{D}$). This is part of the "manifold property" for a polyhedron. Let $F_{\mathcal{D}}$ be the set of faces corresponding to $\mathcal{D}$. The faces freely generate a $\mathbb{Q}$-module of *algebraic 2-chains* denoted $C_2(P)$. Taking the given orientations (with the simplicial boundary mapping $\partial : C_2(P) \to C_1(P)$) we get

$$\partial \sum_{f \in F_{\mathcal{D}}} f \;=\; \sum_{d \in \mathcal{D}} z_d.$$

But if all the edges in $\mathcal{D}$ are paired, then by the definition of the boundary of a 2-chain (and using the "outward" positive orientation), they are paired with opposite signs in $C_1(G)$. Hence $\partial \sum_{f \in F_D} f = 0$, and by simplicial topology F_D must be the 2-skeleton of a polyhedron in 3-space. Similarly $F \setminus F_D$ is the two-skeleton of another polyhedron. But these two polyhedra cannot have any intersection in 3-space, since this would violate the manifold property of a polyhedron and its boundary. And if they do not intersect, the polyhedron P could not be connected, contradicting the meaning of "polyhedron".

So we have established the existence of an edge e for every subset $\mathcal{D} \subset \mathcal{C}$ with the above exclusive property. From this follows immediately that the set of algebraic cycles

$$W_{\mathcal{D}} = \{z_f\}, \quad f \in F_D$$

is linearly independent over $\mathbb{Q}$.

Choose a face f_O and let $F_O = F \setminus \{f_O\}$ and $\mathcal{D}_O$ be such that $F_O = F_{\mathcal{D}_O}$, i.e., $\mathcal{D}_O = \mathcal{C} \setminus \{c_{f_O}\}$. Then $W_{\mathcal{D}_O}$ has $n_F - 1$ linearly independent elements. By Euler's formula since P is convex,

$$n_F - n_E + n_V = 2.$$

Thus $n_E - n_V + 1 = n_F - 1$, so $W_{\mathcal{D}_O}$ is a spanning set for cycles of P_1 over $\mathbb{Q}$, and by Lemma 1 is also an integral basis for cycles. Let i and j range over the set F_O, suitably ordered. The inner product $w_i \cdot w_j$ on 1-chains is generated by the formula $e_i \cdot e_j = \delta_{ij}$, which is extended by bilinearity. Then by Theorem 1 the intersection form matrix,

$$R_{ij} = (w_i \cdot w_j)$$

has determinant equal to $\Omega(P_1)$.

Next consider the dual polyhedron P^*. We also know that $\Omega(P^*) = \det \tilde{M}(P^*)$ where $\tilde{M}$ is the vertex incidence matrix, less a row and (same) column. Let v_O^* be the vertex of P^* corresponding to the face f_O of P. Then we may write

$$[\tilde{M}]_{ij} \;=\; (v_i^* \cdot v_j^*),$$

where i and j range over $V^* \setminus \{v_O^*\}$ with the ordering induced by that of F_O.

But the meaning of $v_i^* \cdot v_j^*$ is just

(1.4) if $i = j$, to take $+m_i$, where m_i is the multiplicity of edges incident to vertex v_i^*,

(1.5) if $i \neq j$, to take $-m_{ij}$, where m_{ij} is the multiplicity of edges e^* in $E(P^*)$ with $g(e^*) = \{v_i^*, v_j^*\}$.

If $i = j$, consider the edges e_{ik}^* incident to v_i^*. Their total multiplicity is m_i. But in the construction of the dual, each such edge arises from an edge $\hat{e}_{ik}$ of P which is incident to (belongs to the border of) f_i, the dual face. In fact, the union of $\{\hat{e}_{ik}\}$ is the geometric cycle c_i and constitutes the border of $f_i \in F(P)$. It is clear for the corresponding algebraic cycle that $z_i \cdot z_i = m_i$, (the inner product is positive definite).

In case $i \neq j$, the edge-sets $E_i = \{e_{ik}^*\}$ and $E_j = \{e_{jk}^*\}$ lead to geometric cycles in P_1, namely c_i and c_j, which lead in turn to algebraic cycles z_i and z_j. If v_i^* and v_j^* are not adjacent, both $v_i^* \cdot v_j^* = m_{ij}$ and $c_i \cdot c_j$ are 0. If v_i^* and v_j^* are adjacent, that is,

$$E_{ij} = E_i \cap E_j \neq \emptyset,$$

each $e^* \in E_{ij}$ is counted with -1 sign in $v_i^* \cdot v_j^*$ according to (1.5). By the dual construction, the edges $\hat{e}$ dual to e^* constitute the intersection of the two geometric cycles c_i and c_j. But since f_i and f_j are adjacent faces (have a common edge), and are considered as 2-chains, then according to the "right-hand rule" orientation chosen for P, $\hat{e}$ must occur as a generator in z_i with opposite sign from how it occurs in z_j. Thus

$$z_i \cdot z_j = -m_{ij} = v_i^* \cdot v_j^*,$$

and the matrices $\tilde{M}$ and R are identical, for this choice of ordering and orientation. Hence

$$\Omega(P_1^*) = \det R = \det \tilde{M} = \Omega(P_1).$$

This completes the proof of Theorem 0.

Remark. Since the essence of Theorem 1 is the same duality as we have used for polyhedra, one may construct an entirely geometric proof of Theorem 0, without considering orientations and chains. In this sense, Theorem 1 becomes a generalization of Theorem 0 where more algebra is necessarily involved.

Section 2. Graphs with Cyclic Action

An *n-rotational* multigraph is one that has n vertices, and on which the cyclic group $\mathbb{Z}_n$ of integers modulo n acts faithfully and freely as a group of multigraph isomorphisms. The freeness condition means that no vertex or edge remains unchanged by the action of any element of $\mathbb{Z}_n$ except for the identity. One edge e in G is said to be a *rotation* of another edge e' if e' is the image of e under the action of a particular group element in $\mathbb{Z}_n$. Given the particular group action (by graph isomorphisms), given any pair of vertices v, v', and edge e such that $\mathrm{tail}(e) = v$, there is a unique edge e' such that

i) e' is a rotation of e,

ii) $\mathrm{tail}(e') = v'$.

An identical statement applies with "tail" replaced by "head" throughout. It is easy to see that such multigraphs are characterized by a set of non-negative integers $\{a_1, \ldots, a_r\}$ where $r = \lfloor \frac{n}{2} \rfloor$, and where by freeness, a_r is even if n is even. Their meaning is that the vertex labeled 0 (modulo n) has an edge to vertex i with multiplicity a_i.

The sole purpose of this section is to construct a basis of cycles for a connected n-rotational multigraph. This basis can be used for some special spanning tree enumerations, as is done in Section 3. Since those examples could have been worked by themselves, the general construction here appears for completeness.

So let G be a connected, n-rotational multigraph, with m edges and take i_1 to be the smallest index between 1 and r such that

$$a_{i_1} > 0.$$

Let $x_1 = \text{g.c.d.}(i_1, n)$, and $y_1 = n/x_1$. Then decompose the vertices into the following subsets

$$\begin{aligned}
&P_0 = (0, i_1, 2i_1, \ldots, n - i_1) = (0, x_1, \ldots, (y_1 - 1)x_1) \\
&\quad \vdots \\
&P_\kappa = (\kappa, i_1 + \kappa, 2i_1 + \kappa, \ldots, n - i_1 + \kappa) = (\kappa, x_1 + \kappa, \ldots, (y_1 - 1)x_1 + \kappa) \\
&\quad \vdots
\end{aligned}$$

for $0 \leq \kappa < x_1$. A vertex can be represented by $[f, \kappa]$, meaning $fx_1 + \kappa$. In P_0 the "fundamental edge" i_1 connects 0 with $g_1 x_1$, where $g_1 x_1 = i_1$. This edge has multiplicity a_{i_1}, as do corresponding edges in P_κ. Thus all edges of the form $j = kx_1$ are "internal" to some P_κ, depending on the starting vertex. Given an edge $j = gx_1 + \beta$, j connects P_0 and P_β.

Collect these residues $\{\beta\}$ and choose the smallest index i_2 that represents one, say $i_2 = hx_1 + \beta_2$. Let $x_2 = \text{g.c.d.}(\beta_2, x_1), y_2 x_2 = x_1, g_2 x_2 = \beta_2$. We can now view $P_0, \ldots, P_{x_1 - 1}$ as broken into "orbits"

$$\begin{aligned}
P_0^1 &= \{P_0, P_{x_2}, P_{2x_2}, \ldots, P_{(y_2-1)x_2}\} \\
P_1^1 &= \{P_1, P_{x_2+1}, P_{2x_2+1}, \ldots, P_{(y_2-1)x_2+1}\} \\
&\vdots \\
P_{x_2-1}^1 &= \{P_{x_2-1}, P_{2x_2-1}, \ldots, P_{(y_2)x_2-1}\}.
\end{aligned}$$

The fundamental edge β_2 joins P_0^1 with $P_{g_2}^1$. At this point a vertex v can be represented by $[f_1, f_2, \kappa]$ and is simply

$$v = f_1 x_1 + f_2 x_2 + \kappa, \quad 0 \leq \kappa < x_2,$$

in the usual notation. We may continue this process and thereby obtain a sequence of *levels* $\mathcal{P}^0, \mathcal{P}^1, \ldots, \mathcal{P}^s$, such that at each level $\mathcal{P}^\alpha$ is a collection of disjoint subgraphs of G

$$\mathcal{P}^\alpha = \{P_0^\alpha, \ldots, P_{x_\alpha - 1}^\alpha\}.$$

Some characterizing properties that hold of these levels are as follows:

(1) $x_0 = n$, so $\mathcal{P}^0 = \{v_0, \ldots, v_{n-1}\}$, simply the collection of vertices of G,
(2) $\mathcal{P}^s = \{G\}$, the singleton set whose element is the entire connected graph G,
(3) $\bigcup_{j=0}^{x_\alpha - 1} V(P_j^\alpha) = V(G)$, the union of the sets of vertices at any level gives *all* of the vertices of G ,
(4) each P_j^α is a subgraph of $P_{j'}^{\alpha+1}$ for exactly one $0 \le j' \le x_{\alpha+1} - 1$;
(5) thus $x_0 = n$ and $x_s = 1$,with $x_{\alpha+1} | x_\alpha$ for $0 \le \alpha < s$.

The inductive definition of $\mathcal{P}^{\alpha+1}$ is as follows. Any vertex $v \in G$ is contained in exactly one P_j^α, $0 \le j < x_\alpha$. In fact

$$v = f_1 x_1 + f_2 x_2 + \cdots + f_\alpha x_\alpha + j,$$

where $0 \le f_l < y_l$, $y_l x_l = x_{l-1}$ $1 \le l \le \alpha$. All edges which are in the image under rotation of the vertices of the edge

$$(0, \sum_{l=1}^{\alpha} g_l x_l)$$

are to be found in P_j^α for some j. So we consider classes (under rotation) of edges represented by $(0, v)$, and let β be the residue of v modulo x_α. Choose the smallest $\beta > 0$ among all of these edges (smallest as standard integer representative of an element of $\mathbb{Z}_n$). Then let $x_{\alpha+1} = \text{g.c.d.}(\beta, x_\alpha)$, $\quad y_{\alpha+1} x_{\alpha+1} = x_\alpha$. By definition we take

$$V(P_0^{\alpha+1}) = \bigcup_{k=0}^{y_{\alpha+1}-1} V(P_{k x_{\alpha+1}}^\alpha)$$
$$V(P_j^{\alpha+1}) = \bigcup_{k=0}^{y_{\alpha+1}-1} V(P_{k x_{\alpha+1}+j}^\alpha)$$

where $0 \le j < x_{\alpha+1}$.

The internal edges of $P_j^{\alpha+1}$ consist of

(1) all internal edges of every subgraph $P_{k x_{\alpha+1}+j}^\alpha$,
(2) all edges in G incident to vertices in $P_{k_1}^\alpha$ and $P_{k_2}^\alpha$ with $k_1 \ne k_2$, which are rotations of an edge $(0, v)$ where $v \equiv 0 \bmod x_{\alpha+1}$.

Equivalent to condition (2) is to have v congruent to 0 modulo β, since β was a minimal representative. Finally, a particular choice of a representative for the rotational class of edges in (2) is made, called $i_{\alpha+1}$, the *principal* $\alpha + 1$-*connector*. Hence $i_{\alpha+1} \equiv 0 \bmod x_{\alpha+1}$. This completes the description of the subgraphs $\{P_j^\alpha\}$.

Next we associate to each level $\mathcal{P}^\alpha$ a set of cycles Y^α. At level $\mathcal{P}^0$, there are no edges in P_j^0, so there are no non-zero cycles and $Y^0 = \emptyset$. At level $\mathcal{P}^{\alpha+1}$ we form the collection of cycles Y^α consisting of

(1) all cycles belonging to Y^α,
(2) new cycles to be added in.

Finally, Y^s will be a set of cycles for $\mathcal{P}^s$, and we will show that Y^s is a $\mathbb{Q}$-basis (basis over the rational numbers) for $C^1(G)$.

In each case the cycles in $Y^{\alpha+1}$ will be contained in $P_0^{\alpha+1}$. That is, edges belonging non-trivially to a cycle $\mathbf{z} \in Y^{\alpha+1}$ will belong to $P_0^{\alpha+1}$ and not to any other $P_j^{\alpha+1}$, $1 \leq j < x_{\alpha+1}$.

The new cycles from $P_0^{\alpha+1}$ consists of three different types:

a) boxes,
b) triangles,
c) lassos.

The sets of boxes, triangles, and lassos are denoted $B^{\alpha+1}, T^{\alpha+1}, L^{\alpha+1}$ respectively. We make use of a canonical orientation on the multigraph G that is induced by the labeling of the vertices $\{v\} = \{0, 1, \ldots, n-1\}$, and the construction of the P_j^{α}. Namely, a connecting edge between $P_{k_1}^{\alpha}$ and $P_{k_2}^{\alpha}$ is consistently oriented (the former subgraph contains the tail, the latter contains the head), if $k_2 - k_1 < \dfrac{x_\alpha}{2} = r_\alpha$, with arithmetic being done modulo x_α. If $k_2 - k_1 = r_\alpha$, then there will be an even number of this type of edge: assign a positive orientation to half of them and a negative orientation to the other half.

For the construction of the *boxes*, consider the principal $\alpha+1$-connector $i_{\alpha+1}$, based at vertex 0 ("the origin"), which clearly lies in P_0^{α}. The head of i_α then lies in P_β^{α} in our given notation. Let $A = P_{k\beta}^{\alpha}$, $A' = P_{(k+1)\beta}^{\alpha}$, where lower indices are evaluated modulo x_α, and where $0 \leq k < y_\alpha - 1$. Then, given a vertex $v \in A$, there is a rotation of $i_{\alpha+1}$ whose tail is v, and whose head is a vertex $v' \in A'$.

Now take an edge $e \in A$, whose boundary according to the given "natural" orientation is

$$\partial e = v_2 - v_1.$$

Let d_1 (resp. d_2) be the connecting edge which is a rotation of $i_{\alpha+1}$ whose tail is v_1 (resp. v_2). Then the heads of d_1 and d_2, called v_1' and v_2' respectively, both lie in A'. In A' lies an edge e' which is a "translation" of e by $i_{\alpha+1}$, namely its tail and head are v_1' and v_2' respectively.

Consider the oriented ("algebraic") cycle

$$\mathbf{b}_e = e + d_2 - e' - d_1.$$

Forming such a "box" for each edge $e \in P_{kx_\alpha}^{\alpha}$, $k = 0, \ldots, y_{\alpha_1} - 1$, gives all the boxes necessary. To summarize, to each edge lying in some $P_{k\beta}^{\alpha}$, $k = 0, \ldots, y_\alpha - 2$, there is a box $\mathbf{b}_e$ to be added to the collection of cycles $Y^{\alpha+1}$.

Next we go to the construction in case b), triangles. To any edge e that connects a vertex in $P_{kx_\alpha}^{\alpha}$ with a vertex in some $P_{k'x_\alpha}^{\alpha}$, $k, k' = 0, \ldots, y_\alpha - 1$, we associate a new cycle $\mathbf{t}_e$ called a "triangle". The edge e may be anything except a principal connecting edge $i_{\alpha+1}$ discussed above; it may well be a "replicate" in the multigraph sense, having the same head and tail as $i_{\alpha+1}$. If we rotate e to the origin, it now takes the form $(0, q)$, where

$$q = \sum_{i=1}^{\alpha} \gamma_i x_i + f x_{\alpha+1}, \quad \text{where } f \equiv k' - k \bmod y_{\alpha+1}, \quad 0 \leq \gamma_i < y_i.$$

Suppose that the tail of e is in fact $v_0 \in A \equiv P^{\alpha}_{kx_\alpha}$ and its head is $v_q \in A' \equiv P^{\alpha}_{(k+f)x_\alpha}$. Now the principal connector $i_{\alpha+1}$, when multiplied by the integer $-f$, gives a chain that can be rotated so that its tail is at

$$v_q \in A.$$

The head of this chain u is in A, and we may name it w $(= \sum_{j=1}^{\alpha-1} \delta_j x_j + kx_\alpha)$. But A is by construction a connected subgraph of G, so one may choose a chain a_e with boundary $w - v_0$ that lies entirely in A. In fact a_e can inductively be exhibited as a sum of integer multiples of principal connectors i_γ, $\gamma = 1, \ldots, \alpha$.

This being done, we define the triangle to be the 1-chain

$$\mathbf{t}_e = e + u_e - a_e.$$

The reader may have noticed that we did not include in our definition of "box" the case corresponding to the subgraph $A = P^{\alpha}_{(y_\alpha - 1)\beta}$; one might have expected that the principal connectors

$$i_{\alpha+1} : A \longrightarrow A' = P^{\alpha}_0$$

be used to form boxes. But this would result in a linearly *dependent* (over $\mathbb{Z}$) set of cycles. Instead we form another set of cycles utilizing those "missing" edges that are rotations of $i_{\alpha+1}$.

Starting at a given vertex v_0 in $A_0 = P^{\alpha}_0$, we arrive via $i_{\alpha+1}$ at v_1 in $A_1 = P^{\alpha}_{\beta}$. continuing in this manner gives a sequence of vertices $\{v_k\}$, where $0 \leq k \leq y_{\alpha+1} - 1$, $v_k \in A_k \equiv P^{\alpha}_{k\beta}$, according to

$$\partial i_{\alpha+1} = v_{k+1} - v_k.$$

Repeating this construction once more for $k = y_{\alpha+1} - 1$ gives an edge with tail at v_k and head at $v_0' \in A_0$. Recalling from above that any two vertices in A_0 have a canonical chain c_0, constructed from rotations of principal connectors i_γ, $\gamma \leq \alpha$, such that $\partial c_0 = v_0' - v_0$, we define the lasso based at v_0 to be the chain

$$\ell_{v_0} = (i_{\alpha+1})_{v_0} + (i_{\alpha+1})_{v_1} + \cdots + (i_{\alpha+1})_{v_{y_{\alpha+1}-1}} - c_0.$$

The subscript $v_0, \ldots$ refers to the tail vertex of the given edge. The set of lassos L is now the collection of all ℓ_{v_0} over all $v_0 \in P^{\alpha}_0$.

Consider the totality of cycles found in $Y^s = Y^{s-1} \cup B^s \cup T^s \cup L^s$. In order to show that this set forms a basis of cycles for G over the field $\mathbb{Q}$, it is sufficient to show that

(1) Y^s is a $\mathbb{Q}$-linearly independent set,
(2) the cardinality of Y^s is $m - n + 1$, where m is the total number of edges in G, counting multiplicity.

We set about demonstrating the linear independence, item (1). First of all, Y^0 is empty, hence linearly independent. Assuming that Y^α is $\mathbb{Q}$-linearly independent, suppose that there exists a rational linear combination

$$\Delta \equiv \sum \delta_c c, \quad \delta_c \in \mathbb{Q}, \quad c \in Y^{\alpha+1},$$

such that Δ is 0 as a chain in $C^1(P_0^{\alpha+1})$. Suppose that a *lasso* $\ell \in L^{\alpha+1}$ occurs with non-zero coefficient δ_ℓ in Δ. By construction, some rotated version of the principal connector $i_\alpha : P^\alpha_{(y_{\alpha+1}-1)\beta} \to P_0^\alpha$ occurs non-trivially in ℓ. Inspection of the construction of $Y^{\alpha+2}$ shows that this edge ρ, occurs in no other cycle of this set; hence we have contradicted the assumption that $\Delta = 0$.

Next suppose that there is a triangle $\mathbf{t} \in T^{\alpha+1}$ such that $\delta_{\mathbf{a}} t \neq 0$. Then $\mathbf{t}$ arises in the construction from an edge

$$j : P^\alpha_{kx_{\alpha+1}} \to P^\alpha_{k'x_{\alpha+1}}.$$

The edge j is contained with coefficient ± 1 in the chain $\mathbf{t}$. Furthermore, j occurs in no element of $Y^{\alpha+1}$ besides $\mathbf{t}$. Hence we must have as before that $\delta_t = 0$ for all triangles $\mathbf{t} \in T^{\alpha+1}$.

The third and final case is when a *box* $\mathbf{b} \in B^{\alpha+1}$ has a non-zero coefficient δ_b in Δ. By the construction of boxes, there is then an edge $e \in P^\alpha_{k\beta}$, where $\beta = i_{\alpha+1}$ modulo x_α, such that

$$\mathbf{b} = e + d_2 - e' - d_1.$$

Here d_1, d_2 are rotations of $i_{\alpha+1}$ and e' is a rotation of e lying in $P^\alpha_{(k+1)\beta}$. Now there are other cycles of $B^{\alpha+1}$ that contain d_1 and d_2 non-trivially; these are boxes of the form

a) $\tilde{e} + d_3 - \tilde{e}' - d_2$, or
b) $\tilde{e} + d_1 - \tilde{e}' - d_4$, or
c) $\tilde{e} + d_2 - \tilde{e}' - d_1$.

In the last case c), the edge $\tilde{e}$ is a replicate of e. In taking a linear combination of such chains, a standard argument shows that the only way to have the pertinent connecting edges d_i add up to zero, is to remain with a cycle $p' - p$. Here, $p \in C^1(P^\alpha_{(k+1)\beta})$ and $p' \in C^1(P^\alpha_{k\beta})$ are both cycles. Subsequently, the only way to eliminate p', say is to use boxes (other cycles of Y^α having been previously ruled out for use in the linear combination Δ) that connect to the subgraph $P^\alpha_{(k+2)\beta}$. But this gives rise in Δ to another cycle $p'' \in A_{k+2}$, so we remain with the task of eliminating $p - p''$. The only chance to do this is to repeat our process of adding in more boxes, which finally results in a cycle $q - q'$,

$$q \in C^1(P_0^\alpha), \qquad q' \in C^1(P^\alpha_{(y_{\alpha+1}-1)\beta}).$$

But there are no boxes containing edges connecting P_0^α and $P^\alpha_{(y_{\alpha+1}-1)\beta}$ (except for when $y_{\alpha+1} = 2$ which has already been dealt with), no progress can be made. Thus, the assumption that the box $\mathbf{b}$ is part of a $\mathbb{Q}$-linear dependence in B^α leads to a contradiction.

Therefore a linear dependence Δ among elements of $Y^{\alpha+1}$ can only involve elements of Y^α (no boxes, triangles, or lassos at level $\alpha+1$). But by an induction hypothesis, such a linear dependence does not exist either. Hence Y^s is a linearly independent set of cycles for G.

Next, to settle item (2) above, we enumerate the number of elements in $Y^{\alpha+1}$. Let n_α, m_α be the number of vertices and edges respectively in the subgraph P_0^α. We recall that $\mathrm{card}(Y^0)=0$. But $n_0=1, m_0=0$, so $\mathrm{card}(Y^0)=m_0-n_0+1$, as an initial step for an induction. Also, $n_s=n, m_s=m$, so in proving $\mathrm{card}(Y^s)=m-n+1$, the formula $\mathrm{card}(Y^\alpha)=m_\alpha-n_\alpha+1$ can be taken as an induction hypothesis.

There remains to prove this statement with α replaced by $\alpha+1$, $0\le\alpha<s$. Now in $P_0^{\alpha+1}$ there are $y_{\alpha+1}n_\alpha=m_\alpha$ vertices. There are $y_{\alpha+1}m_\alpha$ "internal" edges inherited from $P_0^\alpha, P_{x_\alpha}^\alpha, \ldots$, and in addition there are $y_{\alpha+1}n_\alpha$ connecting edges arising from principal connectors which are rotations of $(0, i_{\alpha+1})$. Finally, suppose that from each vertex in P_0^α there is a quantity a of other edges incident to some other P_j^α, $j\neq 0$. (By rotational invariance a does not depend on the particular vertex chosen.) This exhausts the different types of edge, and we may write

$$m_{\alpha+1}=y_{\alpha+1}(m_\alpha+n_\alpha+an_\alpha).$$

Now we count the number of elements in the set $Y^{\alpha+1}$. There is one box for each edge of

$$P_{k\beta}^\alpha, \quad k=0,\ldots,y_{\alpha+1}-2.$$

Thus there are $(y_{\alpha+1}-1)m_\alpha$ boxes. For triangles, there is one triangle for each non-principal connector. That gives a total of $an_\alpha y_{\alpha+1}$ triangles. Finally, since if we move the origin of a fundamental connector $(0,i_\alpha)$ within P_0^α, each such rotated connector gives rise to a distinct lasso; hence there are n_α lassos total.

Thus we have

$$\begin{aligned}\mathrm{card}(Y^{\alpha+1})&=\mathrm{card}(Y^\alpha)+\mathrm{card}(B^{\alpha+1})+\mathrm{card}(T^{\alpha+1})+\mathrm{card}(L^{\alpha+1})\\&=m_\alpha-n_\alpha+1+(y_{\alpha+1}-1)m_\alpha+ay_{\alpha+1}n_\alpha+n_\alpha.\end{aligned}$$

But

$$\begin{aligned}m_{\alpha+1}-n_{\alpha+1}+1&=y_{\alpha+1}(m_\alpha+n_\alpha+an_\alpha)-y_{\alpha+1}n_\alpha+1\\&=y_{\alpha+1}m_\alpha+ay_{\alpha+1}n_\alpha+1.\end{aligned}$$

So we do have

$$\mathrm{card}(Y^{\alpha+1})=m_{\alpha+1}-n_{\alpha+1}+1,$$

which was to be proved. In particular, $\mathrm{card}(Y^s)=m^s-n^s+1=m-n+1$, so Y^s has the required number of elements of a $\mathbb{Q}$-basis, and is linearly independent, and hence *is* is $\mathbb{Q}$-basis of $C^1(G)$.

Theorem 2. *The set of cycles written as Y^s is an integral basis of integral cycles for an n-rotationally symmetric multigraph G.*

Proof. By construction, all of the cycles from Y^s are integral, and in fact each edge occurs with coefficient ± 1, so by the Lemma 1, if Y^s is a $\mathbb{Q}$-basis of cycles, it will be a $\mathbb{Z}$-basis of cycles. But verification of the two itemized points above shows that Y^s is a rational basis for the rational cycles of G.

To end the Section we briefly examine the important special case where G_n has its first characterizing number $a_1 > 0$. Clearly then G_n is automatically connected. In fact $x_1 = 1$ and $P_0^1 = G_n$, so the basis $Y^s = Y^1$ has a particularly simple form. Since there are no edges in $P_0^0 = V(G)$, there are no boxes and $B^1 = \emptyset$. Any edge e which is not a rotation of the principal connector $(0,1) \equiv (1)$ gives rise to a triangle $\mathbf{t}_e$. If $\partial e = j - i$, then take

$$\mathbf{t}_e = (1)_i + (1)_{i+1} + \cdots + (1)_{j-1} - e.$$

This is valid too when e is a *replicate* of the principal connector (1).

Besides these 0-level triangles, there is a lasso, namely

$$\ell_n = (1)_0 + (1)_1 + \cdots + (1)_{n-1}.$$

Corollary 1. *According to Theorem 2, the triangles and lasso give an integral cycle basis for G_n as desired.*

Section 3. Computable Integer Sequences

Using Theorem 1, it is possible to obtain recursive formulas for the number of spanning trees for various classes of multigraphs. In this section we apply the "dual" method to certain n-rotational graphs. The results are compared with those obtained in [Vo&Wa]. This comparison leads to two different methods of representing certain sequences of integers. One way consists of a recursive formula utilizing only integer calculations. The other is a "closed-form" expression that requires finer and finer approximation of non-rational numbers (such as roots of unity). The main references for the section are [Sj] and [Vo&Wa], from which we freely quote results.

Let the *algebraic numbers* $\mathbb{A}$ be the algebraic closure of $\mathbb{Q}$. A sequence of integers $\{a_j\}$ is called "algebraically computable" if there are μ constants $h_1, \ldots, h_\mu \in \mathbb{A}$ and γ sequences of algebraic numbers

$$\{b_j^k\}, \quad k = 1, \ldots, \gamma,\ j = 1, \ldots$$

such that the sequence $\{c_f\}$,

$$c_i = h_i, \quad i = 1, \ldots \mu$$
$$c_{\gamma j + k + \mu} = b_j^k$$

has the following properties:

i) $a_i = c_{\gamma(i-1)+1}$,
ii) for $m = \gamma j + k + \mu$, $0 \le k < \gamma$,

$$c_m = g_k(c_{m-1}, \ldots, c_{m-d_k}). \tag{3.1}$$

Here g_k is a polynomial function of d_k variables with coefficients in $\mathbb{A}$ that depends only on k and not otherwise on m.

A similar sequence $\{a_j\}$ is called *integrally computable* if it fulfills the definition of algebraic computability, and in addition all of the constants h_j are integers ($\in \mathbb{Z}$), as are all coefficients of the polynomials g_k.

This is not necessarily the best definition. One might consider replacing the formula (3.1) by

$$\lambda_k(c_m, \dots, c_m - d_k) = 0,$$

where λ_k is an integral multinomial that "happens" always to have an integer solution c_m. For a discussion of this phenomenon, refer to the recent *Mathematical Intelligencer* article by Gale [Ga].

Now consider the graph $G_n(1,2)$ with n vertices and $a_1 = a_2 = 1, a_j = 0,\ j \neq 1, 2$. Select the canonical basis for cycles Y^1 as in Corollary 1. Then the intersection-incidence matrix R_n as in Theorem 1 becomes

$$\begin{bmatrix} C_n & & & 2 \\ & & & \vdots \\ & & & 2 \\ 2 & \dots & 2 & n \end{bmatrix}.$$

Here the first n rows are given by the triangles in T_n, ordered according to the action of a generator of $\mathbb{Z}$. Also, $C_n = \mathbf{circ}(310\dots01)$, the circulant matrix with the indicated first row. The standard (and outstanding) text on circulant matrices is [Da]; for recent applications to communications engineering, see [Gu&Wa]. Now it is seen from [Sj] that $\det R_n = nq_n^2$, where $q_n = q_{n-1} + q_{n-2}$, $n \geq 5$. From this it is clear that $\rho_n = \det R_n$ form terms of an integrally computable sequence.

Looking instead at the vertex-incidence matrix M_n (and $\tilde{M}_n$), one observes that the determinant of $\tilde{M}_n$ equals the product of the eignevalues of M_n except for the 0 eigenvalue. See also [Wo&Fe]. Since M_n is circulant, these eigenvalues are explicitly known in terms of the n-th division point of a circle. This leads quickly to the following formula:

$$\Omega(G_n(1,2)) = n \prod_{\substack{k=1 \\ k\neq r}}^{n-1} (3 + 2\cos\frac{2\pi k}{n}) \tag{3.2}$$

where again $r = \lfloor \frac{n-1}{2} \rfloor$.

This, combined with the above formula for ρ_n, yields

$$q_n = f_n = \prod_{k=1}^{r} (3 + 2\cos\frac{2\pi k}{n}) \tag{3.3}$$

which was first discovered by Bedrosian [Be], also in the context of spanning tree enumeration.

Another algebraic formulation for q_n comes from a general result on page 353 of [Vo&Wa]. It is possible to express $\Omega(G_n)$ in terms of the roots of a polynomial $p(x)$ that depends on $a_1, \dots, a_r$. In case $G = G(1,2)$, one derives

$$p(x) = x^2 + 3x + 1.$$

Letting α, β be the roots of this polynomial ($= \dfrac{-3 \pm \sqrt{5}}{2}$). Then

$$\Omega(G_n) = (-1)^{n+1} \frac{n}{5} (\alpha^n - 1)(\beta^n - 1).$$

This formula arises from equating the two distinct ways of expressing the resultant of the polynomials $p(x)$ and $x^n - 1$. Hence we have

$$q_n^2 = f_n^2 = \frac{(-1)^{n+1}}{5} (\alpha^n - 1)(\beta^n - 1)$$

Substitution for α and β and some manipulation leads to

$$f_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right),$$

the formula of Binet, as on page 148, [Hu].

The right hand side of the Binet formula gives f_n as an algebraically computable sequence where we can take

$$\begin{aligned} c_1 &= \frac{1+\sqrt{5}}{2} \\ c_2 &= \frac{1-\sqrt{5}}{2} \\ c_3 &= \frac{1}{\sqrt{5}} (c_1 - c_2) \\ c_4 &= c_1 \cdot c_2 \\ c_5 &= c_2 \cdot c_2 \\ c_6 &= \frac{1}{\sqrt{5}} (c_4 - c_4) \\ c_7 &= c_1 \cdot c_4 \end{aligned}$$

and so on, where $f_k = c_{3k}$.

On the other hand, the Fibonacci sequence has the integrally computable formulation as follows:

$$\begin{aligned} c_1 &= 1 \\ c_2 &= 0 \\ c_3 &= 1 \\ c_4 &= 1 \\ c_5 &= c_3 + c_4 \\ c_6 &= c_3 \\ &\vdots \end{aligned}$$

Here $\mu = 4$, $f_k = c_{2k-1}$, and the associated functions are

$$\begin{aligned} c_{2k} &= g(c_{2k-1}, \dots) \equiv c_{2k-3} \\ c_{2k+1} &= g(c_{2k}, \dots) \equiv c_{2k-1} + c_{2k}. \end{aligned}$$

We carry through the same process for the graph $G_n(1,3)$, where $a_1 = 1, a_2 = 0, a_3 = 1, \dots$. One derives as in [Sj], using the dual formulation R_n with an expansion of the determinant the formula

$$\Omega(G_n) = \det R_n = \begin{cases} nA_r^2, & n = \text{odd} \\ 2nB_r^2 & n = \text{even}. \end{cases}$$

Certain initial values are given for $\{A_i, B_i, e_i, g_i\}$ for $i \le 5$, where $\{e_i\}$ and $\{g_i\}$ are sequences of integers where the following relations hold.

$$\begin{aligned} e_k &= 4e_{k-1} - 4e_{k-3} + e_{k-4} - 2g_{k-1} + 2g_{k-2} \\ g_k &= 2e_{k-1} - g_{k-1} \\ A_k &= 2e_{k-1} - e_{k-2} + e_{k-4} - 2g_{k-3} \\ B_k &= e_k - e_{k-1}. \end{aligned} \tag{3.4}$$

One has merely to sort out these formulas to obtain an integral computation sequence for $\Omega(G_n)$ or let us say $\{A_r^2, 2B_r^2\}$. But the method of [Vo&Wa] applies here as well, with polynomial

$$p(x) = x^4 + 2x^3 + 4x^2 + 2x + 1,$$

whose roots may be given as $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. In fact we can take $\alpha_1 = \frac{1}{2}(-1 - i + \sqrt{2i - 4})$, and so on (here $i = \sqrt{-1}$). Then we have

$$\Omega(G_n(1,3)) = \frac{n}{10} \prod_{i=1}^{4} (\alpha_i^n - 1). \tag{3.5}$$

This is an algebraic closed-form expression that happens always to be an integer. But $\theta_n = \Omega/n$ is an integrally computable sequence as can be verified using formulas (3.4) above. This fact is not *a priori* self-evident. We have obtained two different computations for θ_n, the one algebraic, the other integral.

We make a final observation concerning θ_n. The polynomial $p(x)$ factors as

$$p(x) = (x^2 + (1+i)x + 1)(x^2 + (1-i)x + 1).$$

Choose a root of the first factor α_1 and a root of the second factor α_3. Then in fact

$$(\alpha_1^n - 1)(\alpha_3^n - 1) = \xi_n + i\eta_n,$$

where ξ_n, η_n are integers. It is unclear whether the sequence

$$\{\xi_n\} = \{3, 4, -3, 4, 13, -20, 3, 64, -93, \\ -36, 333, -380, -387, 1684, -1373, \dots\}$$

is integrally computable. But $\{\xi_n\}$ is clearly algebraically computable since both $\{\xi_n + i\eta_n\}$ and $\{\xi_n - i\eta_n\}$ are (see below).

Since $(\alpha_2 - 1)(\alpha_4 - 1) = \xi_n - i\eta_n$, we get

$$\xi_n^2 + \eta_n^2 = \begin{cases} 10A_r^2 & n = \text{odd} \\ 20B_r^2 & n = \text{even}. \end{cases} \tag{3.6}$$

Thus in particular we have an algebraic method for generating certain solutions of the Diophantine equation

$$a^2 + b^2 = 10c^2.$$

From the lists of $\{\xi_j\}$ and $\{\eta_j\}$ we observe

$$\begin{aligned} 13^2 + (-9)^2 &= 10 \cdot 5^2 \\ 3^2 + 41^2 &= 10 \cdot 13^2 \\ &\vdots \end{aligned}$$

for example. It is well known how *all* such solutions (a, b) are generated, as in Chapter 8 of [LV], but it is of interest to consider why just these solutions arise in the order that they do from the formula (3.5).

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